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Kannan Type Contraction via Interpolation in Bipolar Metric Spaces

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Abstract*: By use of Kannan contraction via interpolation and within the context of Bipolar Metric Spaces, we demonstrate the fixed-point theorem for contravariant mappings. Illustrative example is also provided.*

Keywords: Kannan contraction via interpolation, fixed point, Bipolar Metric Spaces

I. INTRODUCTION

Numerous literary generalizations of the idea of a metric space exist. A bipolar metric space is one of the most recent generalizations. It was developed by Mutlu and Gürdal [8] because distances frequently occur between components of two distinct sets rather than between points of a single set in many real-world applications. In order to formalize these distances, bipolar metrics were developed. The separation between lines and points in a Euclidean space, the separation between sets and points in a metric space, the similarity between a group of students and a set of activities, and the lifetime mean distances between people and places are all examples of distances, and many other examples are some basic examples. For more details, one can see([2]-[6],[9]).

Definition 1 (see [8]). A triple (U, V, ℵ) where U,V are two nonempty sets and $\aleph: \mathrm{U} \times \mathrm{V} \longrightarrow \mathbb{R}_+ = [0, +\infty)$ is a function satisfying the following conditions is a bipolar metric space:

(1) $\aleph(u, v) = 0$ if and only if $u = v$, whenever $(u, v) \in \mathrm{U} \times \mathrm{V}$,

(2) $\aleph(u, v) = \aleph(v, u)$, whenever $u, v \in \mathrm{U} \cap V$,

(3) $\aleph(u_1, v_2) \leq \aleph(u_1, v_1) + \aleph(u_2, v_1) + \aleph(u_2, v_2)$, whenever $(u_1, v_1), (u_2, v_2) \in \mathrm{U} \times \mathrm{V}$.

The pair (U, V) is called a Bipolar Metric.

Definition 2 (see [8]). Let (U_1, V_1) and (U_2, V_2) be pairs of sets and given a function $\mathrm{Y}: U_1 \cup Y_1 \longrightarrow U_2 \cup V_2$.

(1) If $\mathrm{Y}(U_1) \subseteq V_2$ and $\mathrm{Y}(V_1) \subseteq U_2$, we call Y a contravariant map from (U_1, V_1) to (U_2, V_2) and denote this withY: $(U_1, V_1)\ X\ (U_2, V_2)$.

(2) Moreover, if $\aleph_1$ and $\aleph_2$ are bipolar metrics on (U_1, V_1) and (U_2, V_2), respectively, then the notation $\mathrm{Y}: (U_1, V_1, \aleph_1)$ ⤨ $(U_2, V_2, \aleph_2)$ denotes a contravariant map from $(U_1, V_1, \aleph_1)$ to $(U_2, V_2, \aleph_2)$.

Definition 3 (see [8]). Let $(\mathrm{U}, \mathrm{V}, \aleph)$ be a bipolar metric space.

(1) We have

$$\begin{aligned} U &= \text{set of left points;} \\ V &= \text{set of right points;} \\ U \cap V &= \text{set of central points.} \end{aligned}$$

In particular; if $U \cap V = \emptyset$, the space is called disjoint, and otherwise it is called joint. Unless otherwise stated, we shall work with joint spaces.

(2) A sequence (u_n) on the set U is called a left sequence and a sequence (v_n) on V is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.

(3) A sequence (u_n) is said to be convergent to a point u, if and only if (u_n) is a left sequence, $\lim_{n\to\infty}\aleph(u_n,u)=0$ and $u\in V$, or (u_n) is a right sequence, $\lim_{n\to\infty}\aleph(u,u_n)=0$ and $u\in U$.
(4) A bisequence (u_n,v_n) on $(\mathrm{U},\mathrm{V},\aleph)$ is a sequence on the set $\mathrm{U}\times\mathrm{V}$. Furthermore, if the sequences (u_n) and (v_n) are convergent, then the bisequence (u_n,v_n) is said to be convergent. In addition, if (u_n) and (v_n) converge to a common point $t\in U\cap V$, then (u_n,v_n) is called biconvergent.
(5) A bisequence (u_n,v_n) is a Cauchy bisequence, if $\lim_{n\to\infty,m\to\infty}\aleph(u_n,v_m)=0$.
Remark 1 (see [8]). In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

Definition 4 (see [8]). A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Proposition 1 (see [8]). If a central point is a limit of a sequence, then this sequence has a unique limit.
Example 1 (see [8]). Let U be the class of all singleton subsets of $\mathbb{R}$ and V be the class of all nonempty compact subsets of $\mathbb{R}$. We define $\aleph: U\times V\longrightarrow\mathbb{R}_+$as

$$\aleph(u,A)=|u-\inf(A)|+|u-\sup(A)|.$$

Then, the triple $(U,V,\aleph)$ is a complete bipolar metric space.

Definition 5 (see [8]). A covariant or a contravariant map Y from the bipolar metric space $(U_1,V_1,\aleph_1)$ to the bipolar metric space $(U_2,V_2,\aleph_2)$ is continuous, if and only if $(u_n)\longrightarrow v$ on $(U_1,V_1,\aleph_1)$ implies $\left(\mathrm{Y}(u_n)\right)\longrightarrow\mathrm{Y}(v)$ on $(U_2,V_2,\aleph_2)$.

Proposition 2 (see [8]). If the point to which a covariant or contravariant map Y is left and right continuous is central point, then the map Y is continuous at this point.

In 1968, Kannan developed an intriguing form of contraction mapping that has a fixed point and is not continuous [5]. According to Kannan's theorem, if U is a complete metric space and Y : U → U is a self-mapping, then d (Y u, Yv)≤ [d(u, Y u) + d(v, Yv)] must be true for every u, v ∈ U where [0, 1/2). If so, Y has a unique fixed point. Given that U is complete, Kannan was able to demonstrate that f has a unique fixed point if it is a mapping of a Kannan contraction. The term "interpolative contraction" was recently suggested by Karapnar [6] as an extension of the well-known Kannan contraction in metric space.
Let (U, d) be a metric space. A self-mapping Y: U → U is said to be an interpolative Kannan type contraction if there exist a constant λ ∈ [0, 1) and α ∈ (0, 1) such that
$d(\mathrm{Y}u,\mathrm{Y}v)\le\lambda[d(u,\mathrm{Y}u)]^{\alpha}\cdot[d(v,\mathrm{Y}v)]^{1-\alpha}$.

II. MAIN RESULT

In this section, we prove the above result of Kannan type contraction via interpolation in setting of bipolar metric spaces, as follows.

Definition 2.1. Let $(\mathrm{U},\mathrm{V},\aleph)$ be a bipolar metric space. We say that contravariant map $\mathrm{Y}:(\mathrm{U},\mathrm{V},\aleph))\rightleftarrows(\mathrm{U},\mathrm{V},\aleph)$ is an interpolative Kannan type contraction, if there exist a constant $\lambda\in[0,1)$ and $\alpha\in(0,1)$ such that
$\aleph(\mathrm{Y}u,\mathrm{Y}v)\le\lambda[\aleph(u,\mathrm{Y}u)]^{\alpha}\cdot[\aleph(v,\mathrm{Y}v)]^{1-\alpha}$ whenever $(u,v)\in U\times V$ with $u\neq\mathrm{Y}u$.

Theorem 1 (Kannan contraction). Let $(\mathrm{U},\mathrm{V},\aleph)$ be a complete bipolar metric space and $\mathrm{Y}:(\mathrm{U},\mathrm{V},\aleph)\rightleftarrows(\mathrm{U},\mathrm{V},\aleph)$ be a interpolative Kannan type contraction. Then the map $\mathrm{Y}:U\cup V\longrightarrow U\cup V$ has a fixed point.
Proof. Let $u_0\in U$; for each nonnegative integer n, we define $v_n=\mathrm{Y}(u_n)$ and $u_{n+1}=\mathrm{Y}(v_n)$. Then, we have

$$\aleph(u_n,v_n)=\aleph(\mathrm{Y}v_{n-1},\mathrm{Y}u_n)=\aleph(\mathrm{Y}u_n,\mathrm{Y}v_{n-1})\le\lambda[\aleph(u_n,\mathrm{Y}u_n)]^{\alpha}\cdot[\aleph(v_{n-1},\mathrm{Y}v_{n-1})]^{1-\alpha}$$
$$\le\lambda[\aleph(u_n,v_n)]^{\alpha}\cdot[\aleph(v_{n-1},u_n)]^{1-\alpha}$$

which yields that

$$[\aleph(u_n,v_n)]^{1-\alpha}\le\lambda[\aleph(v_{n-1},u_n)]^{1-\alpha}$$

for all integers $n \geq 1$. Then,

$\aleph(u_n, v_n) \leq \lambda\aleph(u_n, v_{n-1})$ (1)

$$\begin{aligned}\aleph(u_n, v_{n-1}) &= \aleph(\Upsilon v_{n-1}, \Upsilon u_{n-1}) \\ &\leq \lambda[\aleph(u_{n-1}, \Upsilon u_{n-1})]^{\alpha} \cdot [\aleph(v_{n-1}, \Upsilon v_{n-1})]^{1-\alpha} \\ &= \lambda[\aleph(u_{n-1}, v_{n-1})]^{\alpha} \cdot [\aleph(v_{n-1}, u_n)]^{1-\alpha}\end{aligned}$$

so that

$\aleph(u_n, v_{n-1}) \leq \lambda\, \aleph(u_{n-1}, v_{n-1})$. (2)

we see that $\lambda < 1$

Moreover, it is easy to see that

$$\aleph(u_n, v_n) \leq \lambda^{2n}\aleph(u_0, v_0),$$
$$\aleph(u_n, v_{n-1}) \leq \lambda^{2n-1}\aleph(u_0, v_0).$$

Hence, for all positive integers m and n, we have

(1) If $m > n$,

$$\begin{aligned}\aleph(u_n, v_m) &\leq \aleph(u_n, v_n) + \aleph(u_{n+1}, v_n) + \aleph(u_{n+1}, v_m) \\ &\leq (\lambda^{2n} + \lambda^{2n+1})\aleph(u_0, v_0) + \aleph(u_{n+1}, v_m) \\ &\vdots \\ &\leq (\lambda^{2n} + \lambda^{2n+1} + \cdots + \lambda^{2m})\aleph(u_0, v_0).\end{aligned}$$

(2) If $m < n$,

$$\begin{aligned}\aleph(u_n, v_m) \leq\ & \aleph(u_{m+1}, v_m) + \aleph(u_{m+1}, v_{m+1}) + \aleph(u_n, v_{m+1}) \\ \leq\ & (\lambda^{2m+1} + \lambda^{2m+2})\aleph(u_0, v_0) + \aleph(u_n, v_{m+1}) \\ & \vdots \\ \leq\ & (\lambda^{2m+1} + \lambda^{2m+2} + \lambda^{2m+1} + \cdots \lambda^{2n})\aleph(u_0, v_0) \\ & + \aleph(u_n, v_n) \\ \leq\ & (\lambda^{2m+1} + \lambda^{2m+2} + \lambda^{2m+1} + \cdots \lambda^{2n} + \lambda^{2n}) \\ & \aleph(u_0, v_0).\end{aligned}$$

Since $\lambda < 1$, this means that $\aleph(u_n, v_m)$ can be made arbitrarily small by larger m and n, and hence (u_n, v_m) is a Cauchy bisequence. Since $(U, V, \aleph)$ is complete, (u_n, v_m) is convergent, and in fact biconvergent, since it is a convergent Cauchy bisequence. Let u be the point to which (u_n, v_m) biconverges. Then, $(u_n) \longrightarrow u, (v_n) \longrightarrow u$, and $u \in U \cap V$. Also, $(v_n) = (\Upsilon u_n) \longrightarrow \Upsilon u$. Since (v_n) has a limit in $U \cap V$, this limit is unique.

Hence, $\Upsilon u = u$, and so Υ has a fixed point.

Example 1. Set $z \in U = V = \{0,1,2,3\}$ with the metric $d(u, v) = |u - v|$ for $u \in U, v \in V$. The contravariant $\Upsilon: (U, V, \aleph)\ X(U, V, \aleph)$ be defined as $\Upsilon 0 = 0$, $\Upsilon 1 = 1, \Upsilon 2 = \Upsilon 3 = 1$ for all $z \in U \cup V$.Let $u \in U, v \in V \setminus$ Fix (Υ). Then, $(u, v) \in \{(2,3), (3,2), (2,2), (3,3)\}$. Then $(U, V, \aleph)$ is a complete bipolar metric space. Thus, (2) is satisfied for all $\lambda \in [0,1)$ and $\alpha \in (0,1)$. Clearly, both 0 and 1 are fixed points for the self-mapΥ.

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