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On the Solution of Integro-Differential Equation Systems by using AboodhTransform.

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Abstract: *In this work a new integral transform, namely Aboodh transform was applied to solve linear systems of Integra-differential equations with constant coefficients.*

Keywords: Aboodh transform -Systems -Integro-differential equation

I. INTRODUCTION

Many problems of physical interest are described by differential and integral equations with appropriate or boundary conditions. These problems are usually formulated as initial value problem, boundary value problems, or initial – boundary value problem that seem to be mathematically more vigorous and physically realistic in applied and engineering sciences. The Aboodh transform method is very effective for Solution of the response of differential and integral equations and a linear system of differential and integral equations.

In this study, Aboodh transform is applied to integral and integrao-differential equations system which the solution of these equations have a major role in the fields of science and engineering. When a physical system is modeled under the differential sense, if finally gives a differential equation, an integral equation or an integrodifferential equation systems. Recently .Aboodh introduced a new transform and named as Aboodh transform which is defined by:

$$A[f(t),v] = K(v) = \frac{1}{v}\int_0^\infty e^{-tv} f(t)\,dt\,, v\,\epsilon\,(-k_1\,,k_2)$$

Or for a function f □t □□which is of exponential order,

$$|f(t)| < \begin{cases} Me^{-tk_1}\,, t \le 0 \\ Me^{tk_2}\,, t \ge 0 \end{cases}$$

The Aboodh transform, henceforth designated by the operator A[] is defined by
the integral equation.

$$A[f(t)] = K(v) = \frac{1}{v^2}\int_0^\infty e^{-t} f\left(\frac{t}{v}\right) dt\,, -k_1 \le v \le k_2$$

Where M is a real finite number and k_1, k_2can be finite or finite.

Theorem (1-1)

Let K (v) is the Aboodh transform of $f\,(t)$

$$A[f(t)] = K(v) \text{ and } g(t) = \begin{cases} f(t-\tau)\,, t \ge \tau \\ 0\,, t < \tau \end{cases}$$

Then

$$A[g(t)] = e^{-tv} K(v)$$

Proof

$$A[g(t)] = \frac{1}{v}\int_\tau^\infty e^{-tv} f(t-\tau)\,dt$$

Let$t = \lambda + \tau$ we find that:

$$\int_0^\infty \frac{1}{v} e^{-(\lambda+\tau)v} f(\lambda)\,d\lambda = e^{-\tau v}\int_0^\infty \frac{1}{v} e^{-\lambda v}\, f(\lambda)\,d\lambda \;= e^{-tv} K(v)$$

Which is the desired result

The Aboobh transform can certainly treat all problems that are usually treated by the well- known and extensively used Laplace transform.

Indeed as the next theorem shows the Aboodh transform is closely connected with the Laplace transform $F(s)$.

Theorem (1-2)

Let

$$f(t)\epsilon B = \begin{Bmatrix} f(t)|\exists M, k_1, k_2 > 0, such\ that\ | \\ |f(t)| < Me^{|t|k_j}, if\ t\epsilon(-1)^j \times [0,\infty) \end{Bmatrix}$$

With Laplace transform F □s □, Then the Aboodh transform K(v) of *f(t)* is given by

$$K(v) = \frac{1}{v}F(v)\ (1)$$

Proof

Let: $f(\mathrm{t}) \in B\ Then\ for$

$$K(v) = \frac{1}{v^2}\int_0^\infty e^{-t} f\left(\frac{t}{v}\right) dt\ , -k_1 \le v \le k_2$$

Let $w = \frac{t}{v}\ Then\ we\ have$

$$K(v) = \frac{1}{v^2}\int_0^\infty e^{-wv} f(w)\ v\, dw = \frac{1}{v}\int_0^\infty e^{-wv} f(w)\ dw$$

$$= \frac{1}{v}\ F(v)$$

Also we have that *K(1) = F(1)* so that both the ELzaki and Laplace transforms must coincide at $v = s = 1$.

In fact the connection of the Aboodh transform with the Laplace transform goes much deeper, therefore the rules of F and T in (1) can be interchanged by the following corollary.

Corollary (1-3)

Let *f (t) t* having F **and** K for Laplace and Aboodh transforms respectively, then:

$$F(s) = \frac{1}{s}K(s)$$

Proof

This relation can be obtained from (1) by taking$V = S$

The equations (1) and (2) form the duality relation governing these two transforms and may serve as a mean to get one from the other when needed.

Aboodh Transform of Derivatives and Integrals.

Being restatement of the relation (1) will serve as our working definition, since the Laplace transform of sin t is $\frac{1}{1+s^2}$ then view of (1), its. Aboodh transform is

$$A[\sin t] = \frac{1}{v(v^2+1)}$$

this exemplifies the duality between these two transforms.

Theorem (2-1)

Let *F(s)* and $T(v)$ be the Laplace and Aboodh transforms of the derivative of $f(t)$**.**

Then:

$(i)\ K'(v) = v\, K(v) - \frac{1}{v} f(0)$

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$(ii)K^{(n)}(v) = v^n K(v) - \sum_{k=0}^{n-1} \frac{1}{v^{2-n+k}} f^{(k)}(0)$

Where $K^{(n)}(v)$ *and* $F^{(n)}(s)$ are the Aboodh and Laplace transforms of the nth derivative $f^{(n)}(t)$ of the function *f (t)*.

Proof

(i) Since the Laplace transform of the derivatives of *f (t)* is

$$F'(s) = s\,F(s) - f(0)$$

Then

$$K'(v) = \frac{1}{v} F'(v)$$

$$= \frac{1}{v}[vF(v) - f(0)] = F(v) - \frac{1}{v} f(0)$$

$$= vK(v) - \frac{1}{v} f(0)$$

(ii) By definition, the Laplace transform for $f^{(n)}$ (*t*) is given by

$$F^{(n)}(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^{(k)}(0)$$

Therefore

$$F(v) = v^n F(v) - \sum_{k=0}^{n-1} v^{n-(k+1)} f^{(k)}(0)$$

Now, since

$$K^{(k)}(v) = \frac{1}{v} F^{(k)}(v)\,, 0 \le k \le m, we\ have$$

$$K^{(n)}(v) = v^n K(v) - \sum_{k=0}^{n-1} \frac{1}{v^{2-n+k}} f^{(k)}(0)$$

Theorem (2-2)

Let $K'(v)$ *and* $F'(V)$ denote the ELzaki and the Laplace transforms of the definite integral of *f(t)*.

$$h(t) = \int_0^t f(\tau) d\tau\ then$$

$$K'(v)\ A[h(t)] = \frac{1}{v} K(v)$$

Proof

By the definition of Laplace transform $F'(s)$ $L\big(h(t)\big) = \frac{F(s)}{s}$

Hence

$$K'(v) \frac{1}{v} F'(v) = \frac{1}{v}\left[\frac{1}{v} F(v)\right]$$

$$= \frac{1}{v^2} F(v) = \frac{1}{v} K(v)$$

Theorem (2-3) (shift)

Let *f (t)*ϵB with Aboodh transform *K (v)*

Then

$$A[e^{at} f(t)] = \frac{1}{1 - \frac{a}{v}} K\left(\frac{1}{v(1 - \frac{a}{v})}\right)$$

Proof

From definition of Aboodh transform we

$$A[e^{at}f(t)] = \frac{1}{v^2}\int_0^\infty e^{-(1-\frac{a}{v})t} f\left(\frac{t}{v}\right) dt$$

Let w =$-(1-\frac{a}{v})t$ dw=$-(1-\frac{a}{v})dt$

Then

$$\frac{1}{v^2(1-\frac{a}{v})}\int_0^\infty f(\frac{w}{v\left(1-\frac{a}{v}\right)} e^{-w}\, dw = \frac{1}{1-\frac{a}{v}} K\left(\frac{1}{v(1-\frac{a}{v})}\right)$$

Theorem (2-4) (convolution)

Let *f (t)t and g (t)* be defined in *A* having Laplace transforms *F (s)* and *G*(*s*) and Aboodh transforms *M* (v) and *N*(*v*) Then the Aboodh transform of the Convolution of *f* and *g*

$$(f\ * g)(t) = \int_0^\infty f(t) g(t-\tau) d\tau$$

Is given by:

$$A[(f\ * g)(t)] = v\, M(v) N(v)$$

Proof

The Laplace transform of (*f* * *g*) is given by:

L(*f* * *g*) $F(s)G(s)$

By the duality relation (1) we have:

$$A[(f * g\,(t))=v\,\frac{1}{v} L[(f\ * g\,)](t)$$

and since

$$M(v) = \frac{1}{v} F(v)\,, N(v) = \frac{1}{v} G(v)$$

Then

$$A((f * g\,)(t))=\frac{1}{v}[F(v)G(v)]))$$
$$=\frac{1}{v}[vM(v).\,vN(v)]= v\,[M(v).\,N(v)]$$

II. SOLUTION OF SYSTEM OF INTEGRO-DIFFERENTIAL EQUATION

Let us consider the general first order system of Integro-differential equation.

$$\begin{cases} y_1{}' = f(t) + \int_0^t [y_1(x) + y_2(x)]dx \\ y_2{}' = g(t) + \int_0^t [y_1(x) - y_2(x)]dx \end{cases} \qquad (5)$$

With the initial conditions

$$y_1(0) = \alpha\,,\ y_2(0) = \beta\ (6)$$

By using Aboodh transform into eq (5) we have,

$$v\overline{y_1} - \frac{1}{v} y_1(0) = \bar{f}(v) + \frac{1}{v}\overline{y_1} + \frac{1}{v}\overline{y_2}$$

$$v\overline{y_2} - \frac{1}{v} y_2(0) = \bar{g}(v) + \frac{1}{v}\overline{y_2} - \frac{1}{v}\overline{y_1} (7)$$

Where $\overline{y_1}$and $\overline{y_2}$are Aboodh transform of $y_1$and $y_2$respectively.

Substituting Eq(6) into Eq(7) we get

$$\left(1 - \frac{1}{v^2}\right)\overline{y_1} = \frac{\alpha}{v^2} + \frac{\bar{f}(v)}{v} + \frac{\overline{y_2}}{v^2}$$

$$\left(\frac{1}{v^2}\right)\overline{y_1} = \frac{\alpha\beta}{v^2} + \frac{\bar{g}(v)}{v} - \left(1 - \frac{1}{v^2}\right)\overline{y_2}$$

Solve these equations to find $\overline{y_1}$ and $\overline{y_2}$

$$\left(\frac{2}{v^4} - \frac{2}{v^2} + 1\right)\overline{y_1} = \frac{\alpha}{v^2} + \frac{(\beta-\alpha)}{v^4} + \left(1 - \frac{1}{v^2}\right)\bar{f}(v) + \frac{\bar{g}(v)}{v^3} = F(v)$$

And

$$y_1(t) = A^{-1}[f(v)] = G(t)$$

Substituting $y_1(t)$ into eq (5) to find $y_2(t)$

Example (1)

Consider the following system

$$\begin{cases} y_1{}' = t + \int_0^t [y_1(x) + y_2(x)]dx \\ y_2{}' = \frac{1}{12}t^4 - 2t + \int_0^t [(t-x).y_1(x)]dx \end{cases} \quad (8)$$

With the initial conditions

$$y_1(0) = 0\,,\ y_2(0) = 1\ (9)$$

By using Aboodh transform into Eq (8) yields

$$v\,\overline{y_1} - \frac{1}{v}y_1(0) = \frac{1}{v^3} + \frac{1}{v}\overline{y_1} + \frac{1}{v}\overline{y_2}$$

$$y_1{}' = t + \int_0^t [y_1(x) + y_2(x)]dx (10)$$

Substituting Eq(9) into Eq (10) we get

$$\left(1 - \frac{1}{v^2}\right)\overline{y_1} = \frac{1}{v^4} + \frac{\overline{y_2}}{v^2}$$

$$\left(\frac{1}{v^3}\right)y_1 = \frac{2}{v^6} + \frac{2}{v^4} - \frac{1}{v^2} + \overline{y_2}$$

Solve these algebraic equations we find that.

$$\overline{y_1} = \frac{2}{v^4}\ and\ \ y_1(t) = t^2$$

Form the first equation of (8) we have

$$y_1{}' = t + \int_0^t [y_1(x) + y_2(x)]dx = t - \frac{1}{3}t^3$$

Applying Aboodh transform to the last equation, we get

$$\frac{1}{v}\overline{y_2} = \frac{1}{v^3} - \frac{2}{v^5} \leftrightarrow \overline{y_2} = \frac{1}{v^2} - \frac{2}{v^4}$$

And

$$y_2(t) = 1 - t^2$$

Example (2)

Consider the following system.

$$\begin{cases} y_1{}'' = -1 - y_1 + \cos t + \int_0^t [y_2(x)]dx \\ y_2{}'' = -y_2 + \sin t - \int_0^t [y_1(x)]dx \end{cases} \quad (11)$$

With the initial conditions

$$y_1(0) = 1\,,\ y_2(0) = 0\ (12)$$
$$y_1{}'(0) = 0, y_2{}'(0) = 1$$

Solution

Applying Aboodh transform to Eq (11) we get

$$v^2\overline{y_1} - y_1(0) - \frac{1}{v}y'_1(0) = \frac{-1}{v^2} - \overline{y_1} + \frac{1}{1+v^2} + \frac{1}{v}\overline{y_2}$$

$$v^2\overline{y_2} - y_2(0) - \frac{1}{v}y'_2(0) = -\overline{y_2} + \frac{1}{v(1+v^2)} + \frac{1}{v}\overline{y_2} \qquad (13)$$

Substituting Eq(12) into Eq(13) we have,

$$\left(1+\frac{1}{v^2}\right)\overline{y_1} = \frac{1}{v^2} - \frac{1}{v^4} + \frac{1}{v^2(1+v^2)} + \frac{1}{v^3}\overline{y_2}$$

The solution of these equations is

$$\overline{y_1} = \frac{1}{(1+v^2)} \; and \; y_1(t) = cost$$

Substituting $y_1(t)$ into eq (11) we get

$$\int_0^t y_2(x)dx = 1 - cost$$

Take Aboodh transform of two side of this equation we have

$$\frac{1}{v}\overline{y_2} = \frac{1}{v^2} - \frac{1}{1+v^2} \leftrightarrow \overline{y_2} = \frac{1}{v} - \frac{v}{1+v^2}$$

$$\overline{y_2} = \frac{1}{v(1+v^2)} \; and \; y_2(t) = \sin t$$

Example (3)

Consider the following linear volterra type Integro-differential equation system.

$$\begin{cases} y_1{}' = 1 + t + t^2 - y_2(t) + \int_0^t [y_1(x) + y_2(x)]dx \\ y_2{}' = -1 - t + +y_1(t) - \int_0^t [y_1(x) + y_2(x)]dx \end{cases} \qquad (14)$$

With the initial conditions.

$$y_1(0) = 1 \,,\; y_2(0) = -1 \; (15)$$

Solution

By taking Aboodh transform of Eq (14) and making use of the conditions (15) we have.

$$\left(1+\frac{1}{v^2}\right)\overline{y_1} = \frac{1}{v^2} + \frac{1}{v^3} + \frac{1}{v^4} + \frac{2}{v^5} - \frac{1}{v}(1+\frac{1}{v})\overline{y_2}$$

$$\frac{1}{v}\left(1-\frac{1}{v}\right)\overline{y_2} = \frac{1}{v^2} + \frac{1}{v^3} + \frac{1}{v^4} + (1-\frac{1}{v^2})\overline{y_2}$$

Solve this equations to find.

$$\left(1-\frac{1}{v}\right)\left(1+\frac{2}{v^2}\right)\overline{y_1} = \frac{-2}{v^6} + \frac{2}{v^5} + \frac{1}{v^4} + \frac{1}{v^3} - \frac{1}{v^2}$$

$$\overline{y_1} = \frac{1}{v^2}\left[\frac{1}{v} + \frac{1}{1-\frac{1}{v}}\right] = \frac{1}{v^3} + \frac{1}{v^2(1-\frac{1}{v})}$$

$$y_1 = A^{-1}\left[\frac{1}{v^3} + \frac{1}{v^2(1-\frac{1}{v})}\right] = t + e^t$$

Where that A^{-1} is the inverse Aboodh transform.
Substituting y_1 into equation (14) we get

$$\overline{y_2} = \frac{-1}{2}t^2 + \int_0^t y_2(x)dx$$

Applying Aboodh transform to this equation, we get.

$$v\overline{y_2} + \frac{1}{v} = \frac{-1}{v^4} + \frac{1}{v}\overline{y_2}$$

$$\left(1 - \frac{1}{v^2}\right)\overline{y_2} = \frac{-1}{v^2} - \frac{1}{v^5}$$

$$\overline{y_2} = \left(\frac{1}{v^2}\left(\frac{1}{v} - \frac{1}{(1-\frac{1}{v})}\right)\right)$$

Then

$$y_2 = A^{-1}\left[\frac{1}{v^3} - \frac{1}{v^2(1-\frac{1}{v})}\right] = t - e^t$$

III. CONCLUSION

In this paper, the Aboodh transform method for the solution of volterra integral and Integro-differential equation systems is successfully expanded. In the first example, the general system of the first order Integro-differential equation and in the last three examples, Integro-differential equation systems are considered. In observed that the Aboodh transform method is robust and is applicable to various types of Integrodifferential and integral equation systems.

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