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New Transform Iterative Method for Solving some Klein-Gordon Equations

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Abstract: *In this study, we treat some Klein-Gordon equations(KGEs). We propose a novel iterative approach called the Aboodh iterative method (AIM). This method, which clearly depends on the choice of the initial values, is based on the new iteration method (NIM) and the Aboodh transformation. We show that the AIM could be more valid and reliable approach than the NIM. We propose an analytical approximation of a solution for KGEs for which only a few iterations are necessary to obtain a semi-analytical solution without a loss of precision.*

Keywords: Aboodh transform, Novel iterative method, Klein- Gordon equations

I. INTRODUCTION

Klein-Gordon equations are becoming more and more common for the modeling of relevant systems in several fields of applied physics. These equations play a major role not only in mathematics but also in physics. For example, we can find a significant role in quantum mechanics[1-3]. Indeed , the presence of a particle in high potential requires a relativistic description [2].On the basis of this description, we can describe in more detail the motion of such a particle , either by the Klein-Gordon equation , or as a component of the spin value of the particle[3].

The Klein-Gordon equation can be expressed in general possible form by :

$$u_{tt} - ku_{xx} + g(u) = 0 \qquad (1.1)$$

In this paper , we propose an analytical approximation tool developed for these specific equations. Several proposed methods based on analytical approximation are developed over these last years. The multi-step differential reduction method is one of these recent examples of these methods [4,5]. Here, we present a modified NIM by using Aboodh transform to solve the Klein-Gordon problem. The Aboodh transform method established by Aboodh [6]. This transform has been applied to solve some differential equations, with variables coefficients, that cannot be solved by other integrals transforms[7-9]. For example, the Aboodh transform is applied to solve ordinary differential equations [10], partial differential equations[11,12] and integral equations. In addition, Aboodh transform should be combined with some other methods, such as homotopy perturbation[13].We can also combined the Aboodh transform with the Adomian

Decomposition method [14] or the variational iteration method [15] to solve nonlinear physical equations. It should be noted that the Aboodh transform was initially chosen to successfully compete with an older method and more developed like the sumudu method . But so far, we have not proved that Aboodh transform is able to solve problems cannot be solved by Laplace. Here ,we propose a method based on the combination of Aboodh transform with the new iterative method (NIM) invented by Daftardar- Gejji and Jafari in [16].The goal, is to obtain a method more precision and having a speed convergence. In numerical section, a particular interest is sharpened to the Klein-Gordon equations.

1.1 Aboodh Transform

Considering the following Aboodh transform of a function $h(x,t)$ with respect to t

$$A[h(x,t)] = A(x,s) = \frac{1}{s}\int_0^\infty h(x,t)\exp(-st)\,dt \qquad (2.1)$$

Where s is a complex number t> 0 $and\ s \in [s_1, s_2]$

Using integration by -part, we can obtain the Aboodh transform of partial derivatives

$$A\left(\frac{\partial h(x,t)}{\partial t}\right) = s\,A(x,s) - \frac{1}{s}h(x,0)$$

Where h is a piecewise continuous and is of exponential order .

We have $A\left(\frac{\partial h}{\partial x}\right) = \frac{d}{dx}[A(x,s)]\ Then$

$$A\left(\frac{\partial^2 h(x,t)}{\partial t^2}\right) = s^2 A(x,s) - h(x,0) - \frac{1}{s}\frac{\partial h(x,t)}{\partial t}(x,0)$$

The result of the nth-partial derivatives is extended by mathematical induction.

1.2 New Iterative Method

The principle of NIM is as follows (see [16,18] for more details):

To find an unknown function u satisfying in the following differential equation

$$u = N(u) + f \ (3.1)$$

Where N is a nonlinear operator and f is a given function. We find a sequences of functions $(u_i)_i$ such that $u = \sum_{n=1}^{\infty} u_i$ Using NIM, the nonlinear operator N is decomposed as follows

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=1}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \qquad (3.2)$$

By Eq (3.1) we obtain

$$N\left(\sum_{i=0}^{\infty} u_i\right) = f + N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=1}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \qquad (3.3)$$

If one considers the following recurrence relation;

$$u_0 = f$$

$$u_1 = N(u_0)$$

$$u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), m\epsilon\mathbb{N} - 0$$

Then

$$u_1 + \cdots + u_{m+1} = N(u_1 + \cdots + u_m)$$

Therefore, the solution of Eq (3.1) can be denoted as the following;

$$u = f + \sum_{i=0}^{\infty} u_i$$

1.3 Novel Aboodh Iterative Method

To illustrate the basic idea of the NAIM, we consider the following equation

$$D_t^n u(x,t) + Ru(x,t) = g(x,t)\ n \in \mathbb{N} \qquad (4.1a)$$

$$u(x,t) = h^0(x), \frac{\partial^k u(x,0)}{\partial t^k} = h^k(x), k \in \{1, \dots\dots, n-1\} \qquad (4.1b)$$

Where $D_t^n = \frac{\partial^n}{\partial t^n}$, R a general nonlinear operator and g as a continuous function.

Aboodh Transform Step

Applying the Aboodh transform to both side of (4.1a), we get

$$A(u(x,t)) - \frac{1}{s^n}\sum_{k=0}^{n-1} \frac{1}{s^{2-n+k}} \frac{\partial^k u(x,0)}{\partial t^k} + \frac{1}{s^n} A\big(Ru(x,t) - g(x,t)\big) = 0$$

Using the initial conditions (4.1b), leads to

$$u(x,t) = A^{-1}\left[\frac{1}{s^n}\sum_{k=0}^{n-1} \frac{1}{s^{2-n+k}} h^k(x)\right] - A^{-1}\left[\frac{1}{s^n} A\big(Ru(x,t) - g(x,t)\big)\right]$$

Thus, we obtain the following typical form

$$u(x,t) = f(x,t) + N[u(x,t)]$$

With

$$f(x,t) = A^{-1}\left[\frac{1}{s^n}\sum_{k=0}^{n-1} \frac{1}{s^{2-n+k}} h^k(x)\right]$$

and

$$N[u(x,t)] = -A^{-1}\left[\frac{1}{s^n} A\big(Ru(x,t) - g(x,t)\big)\right]$$

The function f depends on initial conditions and represents the nonlinear part of equation.

Remark 4.1. Noting the duality between the Aboodh transform A and Laplace transform ℓ

$$A(f)(s) = \frac{1}{s}\ell(f)(s)\ and\ \ell(f)(s) = sA(f)(s)$$

Thus, we simply can obtain the following inversion formula

$$f(t) = \frac{1}{2\pi}\int_{\alpha - i\infty}^{\alpha - i\infty} \frac{1}{s} A(f)(s) \exp\left(\frac{t}{s}\right) ds \qquad (4.3)$$

Iterative method step. Applying the iterative method discussed in Section ‘New iterative method’ to the problem (4.2), we obtain the following algorithm:

$$u_0 = f$$

$$u_1 = N(u_0)$$

$$u_{m+1} = N(u_0 + \dots + u_m) - N(u_0 + \dots + u_{m-1}), m \epsilon \mathbb{N} - 0$$

Which leads to $u = \sum_{i=0}^{\infty} u_i$

II. NUMERICAL EXAMPLES

In this section, some numerical examples are provided to illustrate the ability and precision of NAIM to solve some nonlinear partial differential equations. Besides, it is appropriate that validation of NAIM is accomplished by comparing it with a NIM to make sure of the contribution of the addition of Aboodh transform in the proposed algorithm.

To do so, we will propose to solve homogeneous linear and nonlinear Klein-Gordon equations. Klein-Gordon equation for pionic atom in Coulomb field For begin, we propose an example of the relativistic quantum. Cavalcanti de Oliveira et al. [17] and Durmus et al. [1] have analyzed the movement of a particle in the presence of a dyon with a charged spin 0. They attempted to resolve the Klein-Gordon equation using an analytical method, but the calculation is complicated. To overcome this difficulty, we present a very workable approximation using NIM and NAIM. This is a technique of computation discussed in this example with a secondary aim to find effective solutions that respond to a singular initial value problem. To this end, we propose a radial equation, due to use the spherical coordinate in the analysis of the charged particle-dyon, which leads to the following radial equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(\frac{\alpha}{r^2}+\frac{\beta}{r}+\gamma\right)R(r)=0 \qquad (5.1)$$

subject to the initial conditions

$$R(0) = 0 \; and R'(0) = b. \qquad (5.2)$$

This is a homogenous with a singular point at r = 0. Then Eqs. (5.1),

subject to the initial conditions (5.2), leads to

$$R(r) = -\int_0^r \frac{1}{r^2}\int_0^r (\alpha+\beta r+\gamma r^2)\,R(r)\,dr = N(R)(r)$$

Solution of NIM. Following the description of NIM in Section 'New iterative method' we have $R_{n+1} = N(R_n)$ because N is linear with respect to R. Thus

$$R_0 = br$$

$$R_1 = N(R_0) = \frac{b}{2}\left(\alpha r+\frac{\beta}{3}r^2+\frac{\gamma}{6}r^3\right)$$

$$R_2 = N(R_1) = \frac{b}{2}\left(\frac{\alpha^2}{2}r+\frac{2\beta\alpha}{9}r^2+\frac{\gamma\alpha+6\gamma+2\beta^2}{72}r^3+\frac{\beta\gamma}{40}r^4+\frac{\gamma^2}{180}r^5\right)$$

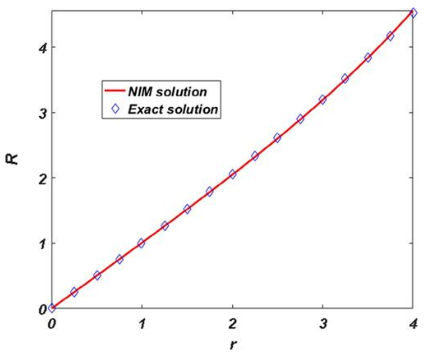

Fig. 1. The NIM solution compared to the exact solution.

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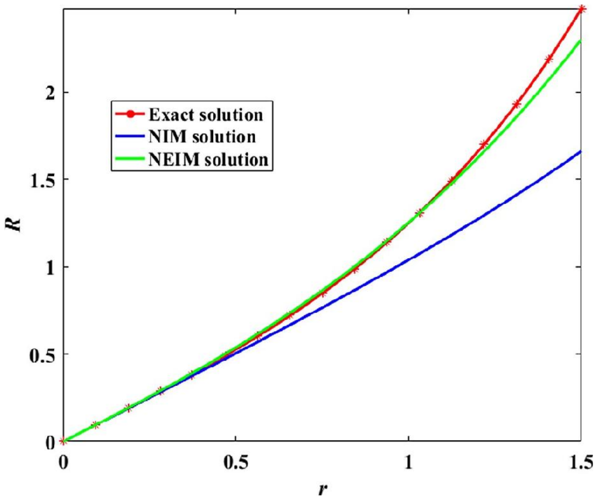
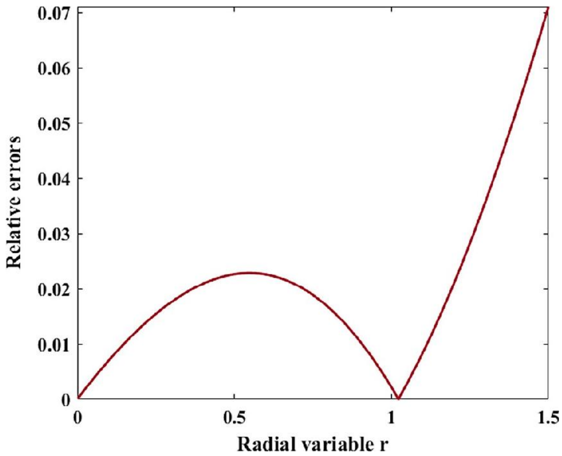


Fig. 2. The NEIM solution compared to the exact solution and the NIM solution.



We can obtain the following approximate solution for the problem (5.1), subject to initial conditions (5.2)

$$R(r) = R_0 + R_1 + R_2 = b\left(\left(\frac{1}{4}\alpha^2 + \frac{\alpha}{2} + 1\right)r + \left(\frac{\beta}{6} + \frac{1}{9}\alpha\beta\right)r^2 + \left(\frac{\beta^2}{72} + \frac{\gamma}{12} + \frac{7\gamma\alpha}{144}\right)r^3 + \frac{\beta\gamma}{80}r^4 + \frac{\gamma^2}{360}r^5\right)$$

For $\alpha = -2, \beta = -0.02$ *and* $\gamma = -0.3$ *and* $b = 1$, we have the following

solution, after carrying out two NIM iterations:

$$R(r) = r + \frac{1}{900}r^2 + \frac{1}{2400}r^3 + \frac{1}{40000}r^4 + \frac{1}{4000}r^5$$

The exact solution is given in [17]

$$R(r) = \exp(-kr)\, r^{sb-1}, F_1(\frac{s_b - nu}{2}, 2s_b, 2kr)$$

where F is the hypergeometric function, $k = -\gamma, u = \frac{\beta}{\gamma}$

and $s_b = 0.5\left(1 + \sqrt{1 - 4\alpha}\right)$

The comparison between the exact solution and that obtained by NIM in Fig. 1 clearly shows that precision has been achieved. Relative error is obtained with a maximal value around 2.5 ×10−3 for r [0 3] and does not exceed 1% if r [0 4]. While this precision may be acceptable with a broad enough interval [0 4], NIM may not be appropriate in problems requiring greater accuracy. NAIM solution. Eq. (5.1) in the following form

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\frac{\alpha}{r^2}+\frac{\beta}{r}+\gamma\right)R(r)=0$$

Applying Aboodh transform and making use of the initial conditions

$$s^2R(s)-\frac{1}{s}b+A\left(\frac{2}{r}\frac{dR}{dr}+\left(\frac{\alpha}{r^2}+\frac{\beta}{r}+\gamma\right)R(r)\right)=0$$

Then

$$R=A^{-1}\left(\frac{b}{s^3}\right)-A^{-1}\left[\frac{1}{s^2}A\left(\frac{2}{r}\frac{dR}{dr}+\left(\frac{\alpha}{r^2}+\frac{\beta}{r}+\gamma\right)R(r)\right)\right]$$

We put

$$N(R)=-A^{-1}\left[\frac{1}{s^2}A\left(\frac{2}{r}\frac{dR}{dr}+\left(\frac{\alpha}{r^2}+\frac{\beta}{r}+\gamma\right)R(r)\right)\right]$$

Note that in the previous expression except for $\alpha=-2$ we canfinalize the calculation using the Aboodh transform because a singular term makes this calculation difficult.

$$R_0=br$$

$$R_1=N(R_0)=-A^{-1}\left[\frac{1}{s^2}A(\beta b+\gamma br)\right]=-b\left(\frac{\beta}{2}r^2+\frac{\gamma}{6}r^3\right)$$

$$R_2(r)=N(R_1)=-b\left(\frac{\beta}{2}r^2+\frac{4\gamma+3\beta^2}{36}r^3+\frac{\beta\gamma}{18}r^4+\frac{\gamma^2}{120}r^5\right)$$

With three-step NAIM we obtain the following solution

$$R(r)=-b\left(\frac{1}{5400}\gamma^3r^7+\frac{23}{10800}\beta\gamma^2r^6+\frac{\gamma(25\beta^2+2\gamma)}{3600}r^5+\frac{\beta(3\beta^2+10\gamma)}{432}r^4+\frac{(3\beta^2+7\gamma)}{54}r^3+\frac{1}{2}\beta r^2-r\right)$$

Using the following value of physical parameters $\alpha=-2$; $\beta=-0.1$ and $\gamma=-1.6$ then $k=8/5$; $nu=1/16$ and $\frac{1}{s}b=2$.

The exact solution is given by

$$R(r)=\exp\left(\frac{-8}{5}r\right)r\ hypergeom\left(\frac{63}{32},4,\frac{16}{5}r\right)$$

With three iterations NAIM we obtain

$$R\approx\frac{-59}{45000}r^5-\frac{1597}{432000}r^4+\frac{1117}{5400}r^3+\frac{1}{20}r^2+r$$

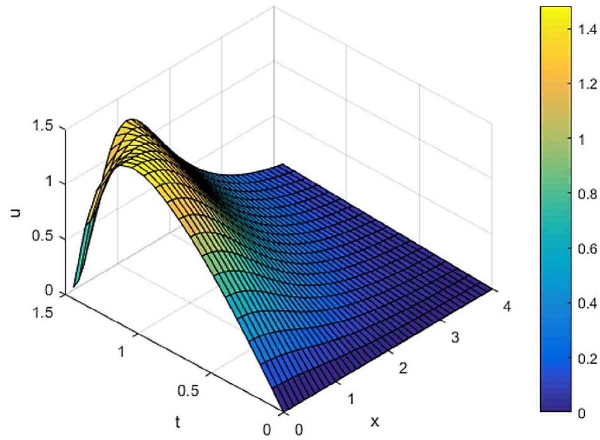


Fig. 4. The solution u of a double-sine-Gordon equation.

In Fig. 2, on the one hand, we compare the results of three-iteration NAIM to the analytical solution. On the other, we raise the number of iterations of NIM until we reached 6. It is clear that the Aboodh transform reduced the duration of calculation and improved precision Fig. 3.

III. THE DOUBLE-SINE-GORDON EQUATION

In the modeling of particle physics, spin wave, quantum field theory and kink dynamics [20,19], double-sine-Gordon equations play an imperative role, particularly, multiple-sine-Gordon equations [21]. Many researchers have proposed different methods of analysis for different sine-Gordon equations. For example, Bin et al. [24] have proposed closed form solutions for this type of equation. Wazwaz [22] has described techniques to solve another form called the double-sine-cosine-Gordon. Most of these authors have suggested techniques assuming that the solution obtained is a wave-solution given physical importance. However, excluding the hypothesis of a wave-solution, does not simplify the search for an analytical solution. Here, we will consider a semianalytical solution using the NAIM of the following equation

$$u_{tt} - ku_{xx} + \alpha \sin(u) + \beta \sin(2u) = 0 \qquad (6.1)$$

Subject to the initial condition

$$u(0,x) = f(x) \text{ and } u_t(0,x) = g(x) \qquad (6.2)$$

Using Taylor series of the function sine, we can obtain

$$u_{tt} - ku_{xx} + (\alpha + 2\beta)u - \frac{\alpha + 8\beta}{6}u^3 + \frac{\alpha + 32\beta}{120}u^5 = 0 \qquad (6.3)$$

Applying the formula of Aboodh transform (4.2) to Eqs. (6.3) and (6.2),

we obtain

$$u(x,t) = f(x) + tg(x) + N(u)(6.4)$$

Where $N(u) = -A^{-1}\left[\frac{1}{s^2}A\left(-ku_{xx} + (\alpha + 2\beta)u - \frac{\alpha+8\beta}{6}u^3 + \frac{\alpha+32\beta}{120}u^5\right)\right]$

Then $u_0 = f(x) + tg(x) \text{ and } u_1 = N(u_0)$), and so forth, until the end of the algorithm of NIM.

Numerical application. Choosing $\alpha = 0$, $k = 1$, $f(x) = 0$, $g(x) = 2\text{sech}(x)$ and $\beta = \frac{1}{2}$, we obtain the following problem

$$u_{tt} - u_{xx} + \frac{1}{2}\sin(2u) = 0 \qquad (6.5a)$$

$$u(0,x) = 0 \text{ and } u_t(0,x) = 2\text{sech}(x) \qquad (6.5b)$$

Using the expression (6.4), we have

$$u_0 = 2t \text{ sech}(x)$$

$$u_1 = N(u_0) = -\frac{2}{3}sech^3(x)t^3 + \frac{4}{15}sech^3(x)t^5 - \frac{32}{315}sech^5(x)t^7$$

$$u_2 = N(u_0 + u_1) - N(u_0) \approx \left(\frac{2}{5}sech^5(x) - \frac{4}{15}sech^3(x)\right)t^5 + \left(\frac{16}{315}sech^3(x) - \frac{64}{315}sech^5(x)\right)t^7$$

With $N(u) = -A^{-1}\left[\frac{1}{s^2}A\left(-u_{xx} + u - \frac{2}{3}u^3 + \frac{2}{15}u^5\right)\right]$

The second-iteration NAIM gives the solution presented in Fig. 4 and given by

$$u = 2tsech(x) - \frac{2}{3}t^3sech^3(x) + \frac{2}{5}t^5sech^5(x) + \left(\frac{16}{315}sech^3(x) - \frac{96}{315}sech^5(x)\right)t^7$$

In [23], the exact solution for the problem $u_{tt} - u_{xx} + \sin(u) = 0\ u(0,x) = 0$

$$u_t(0,x) = 4\,\text{sech}(x)\,, is\; u(t,x) = 4tan^{-1}(t\,\text{sech}\,(x))$$

with this result, and just we replace the variable u in $2u$ so we can easily verify that

$$u_{exact}(t,x) = 2tan^{-1}(t\,\text{sech}\,(x))$$

is an exact solution of problem (6.5).

By developing the Taylor series with order 7 of the function$u_{exact} - u$

in variable t, we obtain

$$u_{exact} - u = t^7\left(\frac{32}{105}sech^5(x) - \frac{16}{315}sech^3(x) - \frac{2}{7}sech^7(x)\right) + o(t^8)$$

IV. CONCLUSION

In this work, we have analyzed certain Klein-Gordon equations. Using NIM and NAIM, we have shown (example 1) that we can acquire an investigative answer for certain differential equation with a singular initial value. We have shown the applicability of NAIM in finding numerical solutions to various physical problems. In the numerical part studied, we noted that we can reduce the level of calculation difficulties using the Aboodh transform method. Although all examples were of the Klein-Gordon equations, other problems can benefit equally well from NAIM. The results show that NEIM can converge more rapidly than NIM and obtain semi-analytical solutions in fewer steps.

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