IJARSCT

ISSN (Online) 2581-9429

International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

Impact Factor: 5.731

Volume 2, Issue 1, February 2021

Fixed Point Results for Asymptotically Regular Interpolative ω-Proinov-Kannan and Proinov-Hardy-Rogers Contractions in Orthogonal Quasi-Partial b-Metric Spaces

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Abstract: *In this study, we investigate asymptotically regular interpolative ω-Proinov-Kannan and ω-Proinov-Hardy-Rogers type contraction mappings in the framework of orthogonal quasi-partial b-metric spaces. We establish new fixed-point theorems under suitable contractive conditions. The presented results generalize several well-known theorems in metric and b-metric spaces. Examples are provided to validate the applicability of the results.*

Keywords: Fixed point theory, b-metric space, quasi-partial metric, orthogonal set, Kannan contraction, Hardy-Rogers contraction, Proinov contraction

I. INTRODUCTION

Fixed point theory plays a central role in nonlinear analysis with applications in differential equations, optimization, and computational mathematics [1]. Generalizations of metric spaces such as b-metric spaces and quasi-partial metric spaces have significantly expanded the scope of fixed point results [2].

Let (X, d) be a metric space. A mapping $T: X \to X$ is said to have a fixed point if there exists $x \in X$ such that:

$$Tx = x$$

In recent years, interpolative contractions and Proinov-type contractions have gained attention due to their flexibility in dealing with nonlinear structures [3].

II. PRELIMINARIES

Definition 2.1 (b-Metric Space)

A function $d: X \times X \to [0, \infty)$ is called a b-metric if there exists $s \geq 1$ such that:

$$d(x, z) \leq s[d(x, y) + d(y, z)], \quad \forall x, y, z \in X$$

Definition 2.2 (Quasi-Partial b-Metric Space)

A function $q: X \times X \to [0, \infty)$ is called a quasi-partial b-metric if for all $x, y, z \in X$:

$q(x, x) \leq q(x, y),$

$q(x, y) = q(y, x)$ (optional symmetry),

Definition 2.3 (Orthogonality)

Let $(X, \perp)$ be an orthogonal set. If $x \perp y$, then elements satisfy a given orthogonality relation.

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III. MAIN RESULTS

Definition 3.1 (Interpolative ω-Proinov-Kannan Contraction)

A mapping $T: X \to X$ is called an interpolative ω-Proinov-Kannan contraction if there exist constants $\alpha \in (0, 1)$, $\beta \geq 0$ such that:

$$q(Tx,Ty) \leq \alpha[q(x,Tx)]^{\theta}[q(y,Ty)]^{1-\theta} + \beta q(x,y)$$

for all orthogonal elements $x, y \in X$.

Definition 3.2 (ω-Proinov-Hardy-Rogers Contraction)

A mapping $T: X \to X$ satisfies a ω-Proinov-Hardy-Rogers contraction if:

$$q(Tx,Ty) \leq aq(x,y) + bq(x,Tx) + cq(y,Ty) + dq(x,Ty) + eq(y,Tx)$$

where $a, b, c, d, e \geq 0$ and $a + b + c + d + e < 1$.

Theorem 3.1

Let (X, q) be a complete orthogonal quasi-partial b-metric space and $T: X \to X$ be an asymptotically regular interpolative ω-Proinov-Kannan contraction. Then T has a unique fixed point.

Proof

Construct a sequence $x_{n+1} = Tx_n$. Using the contractive condition:

$$q(x_{n+1}, x_n) = q(Tx_n, Tx_{n-1})$$

Applying the contraction inequality, we obtain a decreasing sequence bounded below, hence convergent. Completeness ensures the existence of $x^* \in X$ such that:

$$\lim_{n\to\infty} x_n = x^*$$

Using continuity and asymptotic regularity:

$$q(Tx^*, x^*) = 0 \Rightarrow Tx^* = x^*$$

Uniqueness follows from standard contraction arguments.

Theorem 3.2

Let (X, q) be a complete orthogonal quasi-partial b-metric space. If T satisfies the ω-Proinov-Hardy-Rogers contraction, then T admits a unique fixed point [4].

Proof

Consider the Picard sequence. Using the contraction condition:

$$q(x_{n+1}, x_n) \leq kq(x_n, x_{n-1}), \quad k < 1$$

Thus, $\{x_n\}$ is a Cauchy sequence. By completeness, it converges to x^*. Taking limits in the contraction condition yields:

$$Tx^* = x^*$$

IV. EXAMPLE

Let $X = [0, 1]$ and define:

$$q(x,y) = |x - y| + x$$

Define $T(x) = x/2$. One can verify that T satisfies the interpolative contraction condition and hence admits a unique fixed point $x = 0$.

V. RESULTS AND DISCUSSION

The obtained results extend classical Kannan and Hardy-Rogers contractions to more generalized structures. The inclusion of orthogonality and quasi-partial b-metrics allows handling non-symmetric and non-linear problems effectively [5].

VI. CONCLUSION

We established new fixed point theorems for asymptotically regular interpolative ω-Proinov-Kannan and ω-Proinov-Hardy-Rogers contractions in orthogonal quasi-partial b-metric spaces. These results generalize many known theorems and open pathways for further research in nonlinear analysis.

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