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Wave Equation and Bounded Value for Partial Differential Equation

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Abstract: *Partial differential equation plays an important role in almost every application of mathematics where they provide a natural description of many phenomenon involving change in physical science. The concept of wave equation originated in thermodynamics acoustic and statistical physics during 19th century to describe the heat exchange that occur in thermal processes in thermodynamics system. Since wave equation and bounded value have become two of the most important concept in mathematics. In particular wave equation and bounded value have been playing an increasingly important role in partial differential equation in recent decades.*

Keywords: Wave equation, Thermodynamics acoustic conduction, density, propagation, majorant & equation of wave in vacuum etc.

I. Introduction

The wave equation is an important second order linear partial differential equation for the description of wave as they occur in physics- such as thermodynamics/ mechanical wave (eg water waves, sound wave, seismic wave, light wave). It arises in field like thermodynamics, acoustic, electromagnetic and fluid dynamics.

The wave equation is a hyperbolic partial differential equation. It typically concerns a time variable t, one more spatial variable x_1, x_2, x_3 …………and x_n a scalar function $u= u(x_1, x_2, x_3 \ldots\ldots\ldots\ldots x_n\, t)$ whose values could model for example, the heat wave. The wave equation for is:

$$\frac{\partial^2 u}{\partial t^2} = c\nabla^2 u$$

where ∇^2 is the (spatial) laplacian c is a constant.

Solution of this equation describe propagation of heat wave out from the region at a fixed speed in one or in all spatial directions, as do physical waves from plane or localized sources: the constant C is identified with the propagation speed of wave. This equation is linear. Therefore, the sum of any two solutions is again a solution: in physics this property is called the super position principle. The wave equation alone does not specify a physical solution: a unique solution is usually obtained by setting a problem with further conditions such as initial conditions which prescribes the amplitude and phase of the wave. Another important class of problems occurs in enclosed spaces specified by boundary conditions bounded value for which the solution represented heat wave. The wave equation and modification of it are also found in thermodynamics/ elasticity/quantum mechanics/ plasma physics and general relativity.

II. THE EQUATION OF HEAT

The equation of conduction of heat

When heat flows along an insulated uniform straight rod of thermal conductivity K, density ρ and specific heat c, the temperature u at time t at a distance x from a fixed point of the rod satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho c}{K}\frac{\partial u}{\partial t}.$$

Since *K,P, c* are constants, this can be written in the form

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$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

by a change of time-scale. This is the simplest linear equation of parabolic type with two independent variables. It is called the equation of heat or the equation of diffusion. It has one family of characteristics namely the lines t = constant in the xt-plane.

The simplest problem is that of the infinite rod with a given initial temperature distribution

$$u(x, 0) = f(x).$$

On physical ground, it is obvious that the temperature at any sub- sequent instant is uniquely determined. The problem is to find conditions satisfied by f(x) so that this is true, and to find an explicit formula for u

A formal solution of the equation of heat

If, in

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad .....(1)$$

We put u= XT where X and T are functions of x and t respectively, we have

$$\frac{X''}{X} = \frac{T^{\cdot}}{T}$$

Where dashes and dots denote differentiation with respect to x and t.

Hence

$X^{11} = -a^2 X$, $T = -a^2 T$

where a^2 is the separation constant. Thus we have a solution

$u=\exp(-a^2 (t-t_0) \cos a(x—x_0)$,

where x_0 and t_0 are constants.

In the physical problems of heat conduction, u cannot increase indefinitely with t, so we assume that a is real. A more general solution, valid when $t>t_0$, is

$$u = \int_{-\infty}^{\infty} \exp(-a^2(t-t_0)) \cos a(x-x_0)\, da = \frac{\sqrt{\pi}}{\sqrt{(t-t_0)}} \exp\left\{-\frac{(x-x_0)^2}{4(t-t_0)}\right\}.$$

If $x \neq x_0$, this solution tends to zero as $t \to t_0 + 0$.

Other formal solutions can be obtained by integration. For example,

$$u = \frac{1}{2\sqrt{(\pi t)}} \int_{-\infty}^{\infty} f(\xi) \exp(-\tfrac{1}{4}(x-\xi)^2/t)\, d\xi \qquad .....(2)$$

is a solution valid when t>0. If we put

$$\xi = x + 2\eta\sqrt{t}$$

we obtain

$$u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\eta\sqrt{t}) \exp(-\eta^2)\, d\eta, \qquad .....(3)$$

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when t> 0. The limit of this as $t \to +0$ is f(x). Hence (2) is the formal solution of the initial value problem for the infinite rod.

If f (x) is zero when x<a and when x>b where a<b, the solution (2) becomes

$$u = \frac{1}{2\sqrt{(\pi t)}} \int_a^b f(\xi) \exp\left(-\tfrac{1}{4}(x-\xi)^2/t\right) d\xi.$$

If, in addition, f(x) is positive when a< x< b, u(x,t) is positive when t>0 for all values of x. The effect of an initial non-zero temperature distribution on a finite interval is immediately felt everywhere. This result is quite different from that for the equation of wave motions where an initial disturbance restricted to a finite interval is propagated with a finite velocity.

It is convenient to write

$$k(x,t) = \frac{1}{2\sqrt{(\pi t)}} \exp\left(-\tfrac{1}{4}x^2/t\right),$$

so that the formal solution (2) becomes

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)\, k(x-\xi, t)\, d\xi, \qquad \text{…..(4)}$$

when t> 0. This result can be justified if we assume that f (x) possesses a continuous second derivative and satisfies suitable conditions at infinity to ensure the uniform convergence of the integrals obtained from (3) by differentiation under the sign of integration. But, the result holds under very much less restrictive conditions.

he solution (4) was obtained by integrating a multiple

$$k(x-\xi, t-\tau)$$

along a path in the $\xi\tau$ plane. Another formal solution is

$$u(x,t) = \int_0^t \phi(\tau)\, k(x, t-\tau)\, d\tau, \qquad \text{…..(5)}$$

where t> 0, the upper limit being t since k(x, t - r) is complex when r>t. This solution is an even function of x. Its value when x = 0 is

$$\phi(t) = \frac{2}{\sqrt{\pi}} \frac{d}{dt} \int_0^t u(0,\tau) \frac{d\tau}{\sqrt{(t-\tau)}}.$$

This is Abel's integral equation for 5. If u(0, t) is continuous and vanishes when t = 0, the solution of the integral equation is

$$u(0,t) = \frac{1}{2\sqrt{\pi}} \int_0^t \phi(\tau) \frac{d\tau}{\sqrt{(t-\tau)}}.$$

Thus (5) is a formal solution of the equation of heat in terms of the values taken by the solution when x = 0. Yet another formal solution can be obtained by differentiating the expression on the right of (5) with respect to x. It is a multiple of

$$u(x,t) = \int_0^t \phi(\tau) \frac{x}{t-\tau} k(x, t-\tau)\, d\tau, \qquad \text{…..(6)}$$

t being positive. The expression (6) is an odd function of x. If we make the substitution t - r = $\frac{1}{2}x^2/t$ when x> 0, t> 0, (6) gives

$$u(x,t) = \int_0^t \phi(\tau)\frac{x}{t-\tau}k(x,t-\tau)\,d\tau,$$

The limit of this as x--> + O is ∅ (t) when t> 0; but the limit as x $\to 0$ is - ∅(t), since u(x, t) is an odd function of x.

Use of Cauchy—Kowalewsky theorem

The equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{…..(1)}$$

has, by the Cauchy—Kowalewsky theorem, a unique analytic solution, regular in a neighbourhood of (x_0, t_0), satisfying the conditions

$$u(x_0,t) = \phi(t), \quad u_x(x_0,t) = \psi(t), \qquad \text{…..(2)}$$

provided that Φ(t) and ψ(t) are regular in a neighbourhood of t_0. If, these conditions are satisfied at every point (x_0, t_0) of a finite interval γ of the initial line x = x_0, the problem has a unique solution regular near γ. -

By a change of origin, we may take x_0 and t_0 to be zero. By a repeated differentiation of the differential equation, we can calculate all the derivatives of u with respect to x, and obtain the Taylor series

Where ϕ_n and ψ^n are the nth derivatives of ϕ and ψ

Let

$$u(x,t) = \phi(t) + \psi(t)\frac{x}{1!} + \phi_1(t)\frac{x^2}{2!} + \psi_1(t)\frac{x^3}{3!} + \dots + \phi_n(t)\frac{x^{2n}}{(2n)!} + \psi_n(t)\frac{x^{2n+1}}{(2n+1)!} + \dots \qquad \text{….. (3)}$$

$$\phi(t) = \sum_0^\infty a_n t^n, \qquad \psi(t) = \sum_0^\infty b_n t^n,$$

the series having radii of convergence R_1 and R_2 say. Let

R < Min(R_1, R_2)

Since ∅(t) and φ (t) are analytic functions of the complex variable t regular in (t) ≤ R, we have

$$|a_n| \leqslant \frac{M_1}{R^n}, \qquad |b_n| \leqslant \frac{M_2}{R^n},$$

where M_1 and M_2 are the maxima of the moduli of [∅(t) and φ (t)] respectively on [t] = R. Therefore

$$|a_n| \leqslant \frac{M}{R^n}, \qquad |b_n| \leqslant \frac{M}{R^n},$$

where M = Max (M_1, M_2). It follows that, when [t] <R, the function

$$\Phi(t) = \frac{MR}{R-t}$$

is a majorant for ∅(t) and φ (t), and so

$$\frac{MR\,n!}{(R-t)^{n+1}}$$

is a majorant for.

$$\phi_n(t) \text{ and for } \psi_n(t).$$

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If we substitute in (3) the Taylor series for Ø(t) and φ (t) and their derivatives, we obtain a formal double series S. The corresponding' double series S obtained by replacing Ø(t) and φ (t) by Ø(t) is a majorant of S. Typical terms in S are

$$M \frac{x^{2n}}{(2n)!} \frac{n!}{R^n} {}^{n+1}C_k \left(\frac{t}{R}\right)^k$$

and

$$M \frac{x^{2n+1}}{(2n+1)!} \frac{n!}{R^n} {}^{n+1}C_k \left(\frac{t}{R}\right)^k.$$

The double series S is therefore absolutely convergent for all values of [x] and for all values of [t] less than R. Hence the double series S is absolutely convergent for all values of [x] and for all [t[<R and is uniformly convergent on any bounded closed subset. We can therefore differentiate S, and hence also the series (3), term-by-term. It follows that '(3) satisfies the equation of heat under the given conditions when [t[<R This verifies the result of the Cauchy-Kowalewky theorem when

$$u(0,t) = \phi(t), \quad u_x(0,t) = \psi(t),$$

where Ø(t) and φ (t) are analytic functions of t, regular when [t[<R This has an interesting consequence. When t = 0, u(x, t) is equal to F(x), where

$$F(x) = a_0 + b_0\frac{x}{1!} + 1!\,a_1\frac{x^2}{2!} + 1!\,b_1\frac{x^3}{3!} + 2!\,a_2\frac{x^4}{4!} + 2!\,b_2\frac{x^5}{5!} + \ldots$$

The absolute value of the coefficient of x^{2n} is

$$\left|\frac{n!}{(2n)!}a_n\right| \leqslant \frac{n!}{(2n)!}\frac{M}{R^n}$$

And of x^{2n+1} is

$$\left|\frac{n!}{(2n+1)!}b_n\right| \leqslant \frac{n!}{(2n+1)!}\frac{M}{R^n}.$$

Hence F(x) regarded as a function of a complex variable X, is an integral function

Evidently

$$F(x) \leqslant M(1+x)\sum_0^\infty \frac{n!\,n!}{(2n)!}\frac{x^{2n}}{R^n n!}.$$

If

$$A_n = \frac{n!\,n!}{(2n)!}\frac{1}{(KR)^n}$$

We have

$$\frac{A_{n+1}}{A_n} = \frac{(n+1)^2}{(2n+1)(2n+2)}\frac{1}{KR} < \frac{1}{4KR}.$$

If we choose K so that 4KR. A, $\{A_n\}$ is a decreasing sequence and so $A_n < 1$. Therefore

$$F(x) \leqslant M(1+x)\sum_0^\infty \frac{(Kx^2)^n}{n!} = M(1+x)\exp(Kx^2).$$

In particular, when [x] is large

$$|F(x)| < M\exp(2Kx^2),$$

Since

$$|1+x| \leqslant 1+|x| < \exp(Kx^2).$$

Lastly, if we rearrange the double series S as a power series in t, we obtain

$$u(x,t) = F(x) + \sum_{1}^{\infty} \frac{t^n}{n!} F_{2n}(x);$$

Where F_n (x) is the nth derivative of F(x). this is a solution of the equation of heat under the initial conditions (u(x,0)= F(x)

Boundary Conditions

In the problem of heat conduction in a finite rod, there are, in addition to the initial condition, boundary conditions at the end points of the rod. Similar problems arise in the theory of heat conduction in the plane or in space. Suppose that we have a conducting solid bounded by a closed surface S. The temperature u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}$$

and is given initially everywhere, inside S. There are three possible types of boundary condition on S. -

(i) The temperature may be prescribed on S for all time.

(ii) There may be no flow of heat across S so that $\frac{\partial u}{\partial N}$ vanishes on S.

(iii) If the flux of heat across S is proportional to the difference between the temperature at the surface and the temperature u_0 of the surrounding medium, it is equal to $H(u_0 - u)$ where H is a positive constant. The boundary condition is then

$$K\frac{\partial u}{\partial N} = H(u_0 - u),$$

Where $\frac{\partial}{\partial N}$ is differentiation along the outward normal, and K is a positive constant. We write this as

$$\frac{\partial u}{\partial N} + hu = hu_0,$$

where h is a positive constant.,

If the solid is bounded externally by a closed surface S_1 internally by a closed surface S_2, we could have different types of boundary condition on S_1 and S_2

III. CONCLUSION

On the basis of observation of this work we concluded that the wave equation and bounded value is also useful for use in integral transform, an example due to Tikhonov, the case of continuous initial data, the existence and uniqueness theorem, the equation heat in two and three dimensions, the finite rod and the semi infinite rod etc. Wave equation and bounded value is also important for other field of physical science like acoustic electrodynamics and fluid dynamics etc.

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