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Re-Studying Fixed Points Literature on The Branciari-Bakhtin Spaces

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Abstract: In this study, Kannan-Chatrjee's fixed-point types from the literature are re-examined and re-affirmed in the Branciari-Bakhtin space.

Keywords*:* Banach theory, complete metric spaces, generalized space.
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I. INTRODUCTION

The construction of contraction mappings on the metric space gave rise to Banach [4] contraction, which is now widely applied in many scientific and engineering domains. The development of type metric space is established by reducing or changing the metric conditions. For additional information, (see [8,9,11,16]). Notably, abuse or weakening of specific metric criteria results in the loss of some topological advantages, which complicates the proof of several fixed-point theorems. The authors had to devise novel methods to construct fixed-point theorems to address more specialized applications as a result of these difficulties.

Rectangular metric space was defined by Branciari [6], who also discovered the analog of the Banach contraction principle in this space. Many publications have addressed a fixed point theory in the Branciari-metric spaces (see [2,10,20]). However, since Bakhtin [3] proved the symmetric Banach contraction principle and established a generalization of metric space, a great deal of research has been done on the fixed point theorem or the changeful notion for sole-evaluative and multi-evaluative mappings in Bakhtin-metric space (see [1,5, 12, 13, 14, 18, 19]).

As the re-proving parallel of some fixed point theorems in the literature this paper re-study Kannan Chatterjee's types in the Branciari-Bakhtin-metric space.

II. PRELIMINARIES

This section presents the foundation for our primary findings.

Definition 2.1 [3] Suppose $\mathfrak{A}$ be a non-empty set and $\beth \geq 1$, be a given real number. A function $\sigma: \mathfrak{A} \times \mathfrak{A} \rightarrow [0, \infty)$ is a Bakhtin on $\mathbb{X}$ if for all $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \in \mathfrak{A}$, the following are satisfied:

(i): $\sigma(\mathcal{X}_1, \mathcal{X}_2) = 0$ if and only if $\mathcal{X}_1 = \mathcal{X}_2$;

(ii): $\sigma(\mathcal{X}_1, \mathcal{X}_2) = \sigma(\mathcal{X}_2, \mathcal{X}_1)$;

(iii): $\sigma(\mathcal{X}_1, \mathcal{X}_2) \leq \beth[\sigma(\mathcal{X}_1, \mathcal{X}_3) + \sigma(\mathcal{X}_3, \mathcal{X}_2)]$.

In this case, the pair $(\mathfrak{A}, \sigma)$ is called a Bakhtin space**.**

Definition 2.2 [6] Suppose $\mathfrak{A}$ is a nonempty set. A function $\sigma : \mathfrak{A} \times \mathfrak{A} \rightarrow [0, \infty)$ is a Branciari on $\mathfrak{C}$ if for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{A}$ and all distinct points $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{A}$ each distinct from $\mathcal{X}_1, \mathcal{X}_2$, the following terms are satisfied:

(i): $\sigma(\mathcal{X}_1, \mathcal{X}_2) = 0$ if and only if $\mathcal{X}_1 = \mathcal{X}_2$;

(ii): $\sigma(\mathcal{X}_1, \mathcal{X}_2) = \sigma(\mathcal{X}_2, \mathcal{X}_1)$;

(iii): $\sigma(\mathcal{X}_1, \mathcal{Y}_1) \leq \sigma(\mathcal{X}_1, \mathcal{Y}_2) + \sigma(\mathcal{Y}_2, \mathcal{X}_2) + \sigma(\mathcal{X}_2, \mathcal{Y}_1)$.

Then $(\mathfrak{A}, \sigma)$ is called a Branciari metric space.

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Remark 2.1 [14] In Definition 2.2 if the third term becomes to
(iii): $\sigma(\mathcal{X}_1,\mathcal{Y}_1)\le ב[\sigma(\mathcal{X}_1,\mathcal{Y}_2)+\sigma(\mathcal{Y}_2,\mathcal{X}_2)+\sigma(\mathcal{X}_2,\mathcal{Y}_1)]$. Then $(\mathfrak{A},\sigma)$ is called a Branciari-Bakhtin metric space.
Lemma 2.1 [16] Suppose, $(\mathfrak{A},\sigma)$ be a Branciari-Bakhtin metric spaces with $ב\ge 1$ and suppose $\{\mathcal{X}_i\}$ be a Cauchy sequence in $\mathfrak{C}$ such that $\mathcal{X}_i=\mathcal{X}_j$ when it was $i\neq j$. Then $\{\mathcal{X}_i\}$ be able convergence at most one point.

III. MAIN RESULTS

The parallel of of the Kannan's [15,17] fixed point theorem in the Branciari-Bakhtin metric spaces is the next theorem.
Theorem 3.1 Suppose $(\mathfrak{A},\sigma)$ be a complete Branciari-Bakhtin metric spaces with $ב\ge 1$ and suppose ψ is a self-mapping on $\mathfrak{A}$ for all $\mathcal{X}_1,\mathcal{X}_2\in\mathfrak{A}$ satisfies

$$\sigma(\psi\mathcal{X}_1,\psi\mathcal{X}_2)\le(ב+1)^{-1}[\sigma(\mathcal{X}_1,\psi\,\mathcal{X}_1)+\sigma(\mathcal{X}_2,\psi\mathcal{X}_2)]. \quad (3.1)$$

Then ψ has a unique fixed point.
Proof. Assume that $\mathcal{X}_0\in\mathfrak{A}$, consider the iteration $\psi\mathcal{X}_i=\mathcal{X}_{i+1}$ for all $i\ge 1$. We will prove that $\{\mathcal{X}_i\}$ is a Cauchy sequence, such that $\mathcal{X}_i\neq\mathcal{X}_{i+1}$. Using (3.1), to get

$$\sigma(\mathcal{X}_i,\mathcal{X}_{i+1})=\sigma(\psi\mathcal{X}_{i-1},\psi\mathcal{X}_i)\le(ב+1)^{-1}[\sigma(\mathcal{X}_{i-1},\psi\mathcal{X}_{i-1})+\sigma(\mathcal{X}_i,\psi\mathcal{X}_i)]$$
$$\sigma_i\le(ב+1)^{-1}[\sigma_{i-1}+\sigma_i]$$
$$\sigma_i\le ב^{-1}\sigma_{i-1}$$

Using this procedure i times, we get

$$\sigma_i\le(ב)^{-1}\sigma_0 \quad (3.2)$$

Allowing the assumption that $\mathcal{X}_0$ is not a cyclic point of ψ. Actually, if $\mathcal{X}_0=\mathcal{X}_i$ for all $i\ge 2$, we get

$$\sigma(\mathcal{X}_0,\psi\mathcal{X}_0)=\sigma(\mathcal{X}_i,\psi\mathcal{X}_i)$$
$$\sigma(\mathcal{X}_0,\mathcal{X}_1)=\sigma(\mathcal{X}_i,\psi\mathcal{X}_{i+1})$$
$$\sigma_0=\sigma_i$$
$$\sigma_0\le ב^{-1}\sigma_0,$$

using (3.2). Thus, $\sigma_0=0$, hence $\mathcal{X}_0$ is a fixed point of ψ.
Letting $\mathcal{X}_i\neq\mathcal{X}_j$ for all $i\neq j\in\mathbb{N}$, by (3.1), we get

$$\begin{aligned}\sigma(\mathcal{X}_i,\mathcal{X}_{i+2})=\sigma(\psi\mathcal{X}_{i-1},\psi\mathcal{X}_{i+1})&\le(ב+1)^{-1}[\sigma(\mathcal{X}_{i-1},\psi\mathcal{X}_{i-1})+\sigma(\mathcal{X}_{i+1},\psi\mathcal{X}_{i+1})]\\&\le(ב+1)^{-1}[\sigma_{i-1}+\sigma_{i+1}]\\&\le(ב+1)^{-1}[ב^{(i-1)}\sigma_0+ב^{-(i+1)}\sigma_0]\\&=\frac{ב^{(i-1)}}{ב+1}[1+ב^{-2i}]\sigma_0\\&=\mathfrak{S}ב^{(i-1)}\sigma_0\end{aligned}$$

Thus,

$$\sigma(\mathcal{X}_i,\mathcal{X}_{i+2})\le\mathfrak{S}ב^{(i-1)}\sigma_0. \quad (3.3)$$

Where, $\mathfrak{S}=\frac{[1+ב^{-2i}]}{ב+1}>0$.
Know we will exam for $\sigma(\mathcal{X}_i,\mathcal{X}_{i+k})$, $k\in\mathbb{N}$, so we have two cases:
Case i: Take $k=2p+1$

$$\begin{aligned}\sigma(\mathcal{X}_i,\mathcal{X}_{i+2p+1})&\le ב[\sigma(\mathcal{X}_i,\mathcal{X}_{i+1})+\sigma(\mathcal{X}_{i+1},\mathcal{X}_{i+2})+\sigma(\mathcal{X}_{i+2},\mathcal{X}_{i+2p+1})]\\&\le ב[\sigma_i+\sigma_{i+1}]+ב^2[\sigma(\mathcal{X}_{i+2},\mathcal{X}_{i+3})+\sigma(\mathcal{X}_{i+3},\mathcal{X}_{i+4})+\sigma(\mathcal{X}_{i+4},\mathcal{X}_{i+2p+1})]\\&\le ב[\sigma_i+\sigma_{i+1}]+ב^2[\sigma_{i+2}+\sigma_{i+3}]+ב^3[\sigma_{i+4}+\sigma_{i+5}]+\cdots+ב^p\sigma_{i+2p}\\&\le ב[ב^{-i}\sigma_0+ב^{-(i+1)}\sigma_0]+ב^2[ב^{-(i+2)}\sigma_0+ב^{-(i+3)}\sigma_0]+ב^3[ב^{-(i+4)}\sigma_0+ב^{-(i+5)}\sigma_0]\\&\quad+\cdots+ב^pב^{-(i+2p)}\sigma_0\\&\le בב^{-1}[1+בב^{-2}+ב^2ב^{-4}+\cdots]\sigma_0+בב^{-(i+1)}[1+בב^{-2}+ב^2ב^{-4}+\cdots]\sigma_0\\&\quad\vdots\\&=\frac{1+ב^{-1}}{1-בב^{-2}}בב^{-i}\sigma_0.\end{aligned}$$

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Where, $\beth^{-1} < 1$, thus

$$\sigma(\mathcal{X}_i, \mathcal{X}_{i+2p+1}) \le \frac{1+\beth^{-1}}{1-\beth^{-1}} \beth^{1-i}\sigma_0. \tag{3.4}$$

Case ii: Take $k = 2p$

$$\begin{aligned}
\sigma(\mathcal{X}_i, \mathcal{X}_{i+2p}) &\le \beth[\sigma(\mathcal{X}_i, \mathcal{X}_{i+1}) + \sigma(\mathcal{X}_{i+1}, \mathcal{X}_{i+2}) + \sigma(\mathcal{X}_{i+2}, \mathcal{X}_{i+2p})] \\
&\le \beth[\sigma_i + \sigma_{i+1}] + \beth^2[\sigma(\mathcal{X}_{i+2}, \mathcal{X}_{i+3}) + \sigma(\mathcal{X}_{i+3}, \mathcal{X}_{i+4}) + \sigma(\mathcal{X}_{i+4}, \mathcal{X}_{i+2p})] \\
&\le \beth[\sigma_i + \sigma_{i+1}] + \beth^2[\sigma_{i+2} + \sigma_{i+3}] + \beth^3[\sigma_{i+4} + \sigma_{i+5}] \\
&\qquad + \cdots + \beth^{(p-1)}[\sigma_{2p-4} + \sigma_{2p-3}] + \beth^{(p-1)}\sigma(\mathcal{X}_{i+2p-2}, \mathcal{X}_{i+2p}) \\
&\qquad \vdots \\
&\le \beth\beth^{-1}[1 + \beth\beth^{-2} + \beth^2\beth^{-4} + \cdots]\sigma_0 + \beth\beth^{-(i+1)}[1 + \beth\beth^{-2} + \beth^2\beth^{-4} + \cdots]\sigma_0 \\
&\qquad + \beth^{(p-1)}\,\beth^{(2-i-2p)}\sigma_0 \\
&\le \beth^{(1-i)}[1 + \beth^{-1} + \beth^{-2} + \cdots]\sigma_0 + \beth^{-i}[1 + \beth^{-1} + \beth^{-2} + \cdots]\sigma_0 + \beth^{(1-i-2p)}\sigma_0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sigma(\mathcal{X}_i, \mathcal{X}_{i+2p}) &\le \frac{1+\beth^{-i}}{1-\beth\beth^{-2}}\beth\beth^{-i}\sigma_0 + \beth^{(2-i-2p)}\sigma_0 \\
&\le \frac{1+\beth^{-i}}{1-\beth^{-1}}\beth^{(1-i)}\sigma_0 + \beth^{(2-i-2p)}\sigma_0 \\
&\le \frac{1+\beth^{-i}}{1-\beth^{-1}}\beth^{(1-i)}\sigma_0 + \beth^{(2-i)}\sigma_0. \qquad (3.5)
\end{aligned}$$

Therefore, $\lim_{k\to\infty} \sigma(\mathcal{X}_i, \mathcal{X}_{i+k}) = 0$, for all $k > 0$.

Thus, $\{\mathcal{X}_i\}$ is a Cauchy sequence. Since $(\mathfrak{A}, \sigma)$ is a complete Branciari-Bakhtin metric space then there exist $\mathcal{X} \in \mathfrak{A}$ satisfies $\{\mathcal{X}_i\} \to \mathcal{X}$ *as* $i \to \infty$.

To proofing that $\mathcal{X}$ is a fixed point of ψ,

$$\begin{aligned}
\sigma(\mathcal{X}, \psi\mathcal{X}) &\le \beth[\sigma(\mathcal{X}, \mathcal{X}_i) + \sigma(\mathcal{X}_i, \mathcal{X}_{i+1}) + \sigma(\mathcal{X}_{i+1}, \psi\mathcal{X})] \\
&\le \beth[\sigma(\mathcal{X}, \mathcal{X}_i) + \sigma(\mathcal{X}_i, \mathcal{X}_{i+1}) + \sigma(\psi\mathcal{X}_i, \psi\mathcal{X})] \\
&\le \beth[\sigma(\mathcal{X}, \mathcal{X}_i) + \sigma(\mathcal{X}_i, \mathcal{X}_{i+1}) + (\beth+1)^{-1}[\sigma(\mathcal{X}_i, \psi\mathcal{X}_i) + \sigma(\mathcal{X}, \psi\mathcal{X})] \\
&\le \beth[\sigma(\mathcal{X}, \mathcal{X}_i) + \sigma(\mathcal{X}_i, \mathcal{X}_{i+1}) + (\beth+1)^{-1}[\sigma(\mathcal{X}_i, \mathcal{X}_{i+1}) + \sigma(\mathcal{X}, \psi\mathcal{X})]
\end{aligned}$$

$$\begin{aligned}
(1 - \beth(\beth+1)^{-1})\sigma(\mathcal{X}, \psi\mathcal{X}) &\le \beth[\sigma(\mathcal{X}, \mathcal{X}_i) + \sigma(\mathcal{X}_i, \mathcal{X}_{i+1}) + (\beth+1)^{-1}[\sigma(\mathcal{X}_i, \mathcal{X}_{i+1})] \\
&\le \beth[\sigma(\mathcal{X}, \mathcal{X}_i) + (\beth+2)(\beth+1)^{-1}[\sigma(\mathcal{X}_i, \mathcal{X}_{i+1})].
\end{aligned}$$

Then, $\sigma(\mathcal{X}, \psi\mathcal{X}) = 0$, which proving the existence.

To prove the uniqueness, assume that $\mathcal{Z} \in \mathfrak{A}$ such that $\psi\mathcal{Z} = \mathcal{Z}$,

$$\begin{aligned}
\sigma(\mathcal{X}, \mathcal{Z}) = \sigma(\psi\mathcal{X}, \psi\mathcal{Z}) &\le (\beth+1)^{-1}[\sigma(\mathcal{X}, \psi\,\mathcal{X}) + \sigma(\mathcal{Z}, \psi\mathcal{Z})] \\
&\le 0.
\end{aligned}$$

Hence, $\mathcal{X}$ is a unique fixed point of ψ in $\mathfrak{A}$.

The following result parallels the Chatterjee [7] type in a Branciari Bakhtin space.

Corollary 3.1 Let $(\mathfrak{A}, \sigma)$ be a complete Branciari-Bakhtin metric space with $\gimel > 1$ and let $\psi: \mathfrak{A} \to \mathfrak{A}$ satisfies for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{A}$

$$\sigma(\psi\mathcal{X}_1, \psi\mathcal{X}_2) \le (\beth+1)^{-1}[\sigma(\mathcal{X}_1, \psi\,\mathcal{X}_2) + \sigma(\mathcal{X}_2, \psi\mathcal{X}_1)]. \tag{3.6}$$

Then ψ has a unique fixed point.

Example 3.1 Let that $\mathfrak{A} = \mathcal{X}_1 \cup \mathcal{X}_2$ such that $\mathcal{X}_1 = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$, $\mathcal{X}_2 = [1,2]$. Consider $\sigma: \mathfrak{A} \times \mathfrak{A} \to [0, \infty)$ and $\sigma(\mathcal{Z}_1, \mathcal{Z}_2) = 0$, where $\mathcal{Z}_1 = \mathcal{Z}_2$ and $\sigma(\mathcal{Z}_1, \mathcal{Z}_2) = \sigma(\mathcal{Z}_2, \mathcal{Z}_1)$, $\mathcal{Z}_2, \mathcal{Z}_1 \in \mathfrak{A}$, as

$$\begin{cases} \sigma\left(\frac{1}{3},\frac{1}{2}\right) = \sigma\left(\frac{1}{4},\frac{1}{5}\right) = \frac{3}{100} \\ \sigma\left(\frac{1}{5},\frac{1}{2}\right) = \sigma\left(\frac{1}{3},\frac{1}{4}\right) = \frac{2}{100} \\ \sigma\left(\frac{1}{4},\frac{1}{2}\right) = \sigma\left(\frac{1}{4},\frac{1}{5}\right) = \frac{6}{10} \\ \sigma(\mathcal{Z}_1,\mathcal{Z}_2) = |\mathcal{Z}_1 - \mathcal{Z}_2|^2 \ else \end{cases}$$

Therefore $(\mathfrak{A},\sigma)$ is a complete Branciari-Bakhtin metric space with $\beth = 3$. Define $\psi: \mathfrak{A} \to \mathfrak{A}$ as

$$\psi(\mathcal{Z}) = \begin{cases} \frac{1}{4}, & \mathcal{Z} \in \mathcal{X}_1 \\ \frac{1}{5}, & \mathcal{Z} \in \mathcal{X}_2 \end{cases}$$

Hence, condition of Theorem 3.1 is satisfied and ψ has a unique fixed point $\frac{1}{4}$.

IV. CONCLUSION

This paper re-study some important theorems from the literature and re-proves them on the Branciari-Bakhtin metric space.

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