IJARSCT
ISSN: 2581-9429
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
Volume 6, Issue 2, February 2026
Impact Factor: 8.2

The Cauchy Integral in Complex Analysis

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Abstract: *The study of analytic and harmonic functions forms a fundamental part of complex analysis, with applications ranging from physics to engineering. An analytic function is a complex function that is differentiable at every point in its domain, and a key property of such functions is that their real and imaginary parts are harmonic, satisfying Laplace's equation. This connection provides a powerful link between complex function theory and potential theory. However, the converse is not generally true: not every pair of harmonic functions corresponds to the real and imaginary parts of an analytic function [12,15].This discrepancy arises because analyticity imposes additional conditions, namely the Cauchy–Riemann equations, which harmonics alone may not satisfy. This work investigates the relationship between harmonic functions and analytic functions, exploring the conditions under which the former can be associated with the latter and highlighting cases where the converse fails. By analysing examples, counterexamples, and theoretical constraints, this study aims to clarify the subtle distinction between harmonicity and analyticity, thereby providing deeper insight into the structure of complex functions and their applications [11,14,15].*

Keywords: Analytic function, Harmonic function, Cauchy–Riemann equations, Complex differentiability, Necessary condition, Sufficient condition, Partial derivatives, Continuity

I. INTRODUCTION

In complex analysis, the relationship between analytic functions and harmonic functions plays a central and foundational role [11,14]. A complex-valued function $\mathbf{f(z) = u(x, y) + iv(x, y)}$, defined on a domain in the complex plane, is said to be analytic if it is complex differentiable at every point of that domain. One of the most important consequences of analyticity is that the real and imaginary parts of such a function satisfy Laplace's equation, making them harmonic functions. That is, if f is analytic, then both u and v are harmonic, meaning

$$\nabla^2 u = 0 \quad \text{and} \quad \nabla^2 v = 0$$

This result establishes a strong connection between complex analysis and potential theory, with far-reaching applications in physics, engineering, and geometry[7,14]. Harmonic functions naturally arise in the study of steady-state heat flow, electrostatics, fluid dynamics, and gravitational fields. Analytic functions, therefore, provide a powerful framework for constructing and studying solutions to these physical problems.

Complex analysis provides powerful tools for understanding functions of a complex variable and has wide-ranging applications in mathematics, physics, and engineering. Among its core concepts are analytic functions, which are complex functions that are differentiable in a neighbourhood of every point in their domain. A remarkable consequence of analyticity is that the real and imaginary parts of an analytic function satisfy Laplace's equation and are therefore harmonic functions.

The Cauchy integral and the Cauchy–Riemann equations play a central role in establishing this connection. The Cauchy integral formula not only provides an explicit representation of analytic functions but also leads to profound results such as infinite differentiability, analyticity, and the harmonicity of components. Despite this strong relationship, harmonicity by itself does not ensure analyticity. The existence of a harmonic function does not automatically imply the existence of a harmonic conjugate that satisfies the Cauchy–Riemann equations globally.

DOI: 10.48175/IJARSCT-31210

ISSN: 2581-9429 Volume 6, Issue 2, February 2026 Impact Factor: 8.2

The objective of this paper is to examine the role of the Cauchy integral in complex analysis and to clarify the relationship between analytic and harmonic functions. By discussing theoretical results alongside examples and counterexamples, we aim to emphasise why analyticity is a stronger condition than harmonicity.

However, while analyticity guarantees harmonicity of the real and imaginary parts, the converse statement is not true[12,15]. That is, not every pair of harmonic functions arises as the real and imaginary parts of an analytic function. This distinction highlights a subtle but essential aspect of complex differentiability. For a function to be analytic, its components must not only be harmonic but must also satisfy the Cauchy–Riemann equations, which impose additional compatibility conditions between the partial derivatives of u and v [11,12]. Without these conditions, harmonicity alone is insufficient to ensure analyticity.

The failure of the converse leads to important theoretical questions: under what circumstances can a harmonic function be realised as the real part of an analytic function? When does a harmonic conjugate exist, and when is it unique? These questions depend critically on the properties of the domain, such as simple connectivity, and motivate deeper investigation into the structure of harmonic functions [12,13]. In particular, while every harmonic function on a simply connected domain admits a harmonic conjugate locally, global existence is not always guaranteed.

The purpose of this research is to explore the precise relationship between harmonic and analytic functions, emphasising why harmonicity alone does not imply analyticity. By examining counterexamples, necessary and sufficient conditions, and the role of the Cauchy–Riemann equations, this work aims to clarify the limitations of the converse statement. Understanding this distinction not only deepens insight into complex analysis but also strengthens the conceptual bridge between real and complex function theory.

II. METHODOLOGY

1. Cauchy-Riemann Equations:

Let f: C→ C be a complex-valued function defined by,

$$f(z) = u(x, y) + iv(x, y), \quad \text{where } z = x + iy$$

The function f(z) is said to satisfy the Cauchy–Riemann equations [11,15]at a point (x,y) if the first-order partial derivatives of u and v exist and satisfy:[11,12]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Analytic Function:

A complex-valued function $f(z)$ is said to be **analytic** at a point z_0 if it is complex differentiable at every point in some neighbourhood of z_0 [11,14].

If $f(z)$ is analytic at every point of a domain D⊂C, then $f(z)$ is said to be **analytic in** D[12].

Conditions for a function to be analytic:

Complex differentiability:

$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$

must exist and be the same for all directions of h∈C.

2.Cauchy–Riemann equations:

If $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$

Then at a point, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

3. Continuity of partial derivatives:

If the first partial derivatives of u and v exist, are continuous, and satisfy the Cauchy–Riemann equations in a neighbourhood, then the function is analytic there.

Harmonic Function:

A function $u(x, y)$ is called harmonic in a domain D⊂R^2 if:[13,14]

DOI: 10.48175/IJARSCT-31210

1. u has a continuous second order partial derivative in D, and
2. u satisfies laplace's equation in D:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Theorem: Necessary and Sufficient Conditions for Analyticity

Statement:

Let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined on a domain D⊂C,where $z = x + iy$

If f is analytic at $z_0 = x_0 + iy_0$ then u and v satisfy the Cauchy-Riemann equations at (x_0, y_0) and partial derivatives exist. (Necessary condition)[11,15]

If u and v have continuous first partial derivatives in a neighbourhood of z_0 and satisfy the Cauchy-Riemann equations, then f is analytic at z_0. (Sufficient condition)[12,14]

Proof: part 1(Necessary condition)

Given: f is analytic at $z_0 = x_0 + iy_0$.

Since f is analytic, its derivative $f'(z_0)$=$\lim_{z\to z_0} \frac{f(z_0+h)-f(z_0)}{h}$ exists and is the same in every direction.

Taking the limit along the x-direction gives $f'(z_0) = u_x + iv_x$

Again, taking the limit along the y-direction gives $f'(z_0) = v_y - iu_y$

Since both limits are equal,

$$u_x = v_y \ , \ u_y = -v_x$$

So, partial derivatives exist, and the Cauchy-Riemann equations are satisfied at (x_0, y_0) [11,15].

Part 2(Sufficient condition)

Given: u and v have continuous first partial derivatives in a neighbourhood of z_0 and satisfy the Cauchy-Riemann equations [12].

We have to prove that f is analytic at z_0.

Given the equation, we can write as the form,

$$\Delta f \approx (u_x + iv_x)(\Delta x + i\Delta y)$$

Dividing by $\Delta z = \Delta x + i\Delta y$ and letting $\Delta z \to 0$,

$$\lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z} = u_x + iv_x$$

Thus, the derivative exists at z_0.

So,f is analytic at z_0.

Remark:

Every analytic function is harmonic, but every harmonic function is not analytic.

(i.e Every analytic function is harmonic, but convers is not true)[13,15]

A complex function $f(z) = u(x,y) + iv(x,y)$ is analytic at z_0 iff u and v have continuous first partial derivatives in a neighbourhood of z_0 and satisfy the Cauchy-Riemann equations at (x_0, y_0)**.**

Example:

$f(z) = z^2$

$\Rightarrow$ **Given,** $\mathbf{f(z) = z^2}$

$\mathbf{z^2 = x^2 - y^2 + i2xy}$

$u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$

First, we check the Cauchy-Riemann equation,

$u_x = v_y \quad , \quad u_y = -v_x$

Compute derivatives:

$u_x = 2x$, $u_y = -2y$

$v_x = 2y$, $v_y = 2x$

Check equation:

$u_x = 2x = v_y$, $u_y = -2y = -v_x$

Cauchy-Riemann equations are satisfied everywhere.

So $f(z) = z^2$ is analytic everywhere [11].

Second, we check for harmonicity,

A function is harmonic if $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

For $u = x^2 - y^2$

$u_{xx} = 2, u_{yy} = -2 \quad \Rightarrow u_{xx} + u_{yy} = 0$

For $v = 2xy$

$v_{xx} = 0, v_{yy} = 0 \quad \Rightarrow v_{xx} + v_{yy} = 0$

Both u and v are harmonic

By the above remark, every analytic function is harmonic

$$\boldsymbol{f(z) = \bar{z}}$$

⇒ **Given,** $f(z) = \bar{z}$

$\bar{z} = x - iy$

$\boldsymbol{u(x,y) = x}$ and $\boldsymbol{v(x,y) = -y}$

first we check for harmonicity,

A function is harmonic if $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

For $u = x$

$u_{xx} = 0, u_{yy} = 0 \quad \Rightarrow u_{xx} + u_{yy} = 0$

For $v = -y$

$v_{xx} = 0, v_{yy} = 0 \quad \Rightarrow v_{xx} + v_{yy} = 0$

Both u and v are harmonic[13,14].

Second, **we check the Cauchy-Riemann equation,**

$u_x = v_y$, $u_y = -v_x$

Compute derivatives:

$u_x = 1$, $u_y = 0$

$v_x = 0$, $v_y = -1$

Check equation:

$u_x \neq v_y$, $u_y = -v_x$

The second Cauchy-Riemann equation fails.

So $f(z) = \bar{z}$ is not analytic.

Above remark, **every harmonic function is not analytic.**

III. RESULT

Every analytic function defined on a domain D⊂C is harmonic, but the converse is not true [14,15].

If $f(z) = u(x,y) + iv(x,y)$ is analytic in D, then the real and imaginary parts u and v satisfy the Cauchy–Riemann equations. These equations imply that both u and v satisfy Laplace's equation, $\nabla^2 u=0$ and $\nabla^2 v=0$ and hence are harmonic functions. Therefore, harmonicity is a necessary consequence of analyticity.

The above examples illustrate the fundamental distinction between harmonic and analytic functions. While analyticity implies harmonicity, the converse is false in general. The Cauchy–Riemann equations impose additional compatibility conditions that are not guaranteed by harmonicity alone. The role of the Cauchy integral is crucial in understanding this distinction, as it provides a unifying framework for analyticity, harmonicity, and boundary value problems.

However, harmonicity alone is not sufficient to guarantee analyticity [12,13]. A harmonic function may fail to satisfy the Cauchy–Riemann equations or may not admit a harmonic conjugate on the given domain. Consequently, such a function cannot be expressed as the real or imaginary part of an analytic function. This establishes that the class of harmonic functions is strictly larger than the class of analytic functions.

Thus, analyticity implies harmonicity, but harmonicity does not imply analyticity unless additional conditions, such as the existence of a harmonic conjugate in a simply connected domain, are satisfied.

IV. CONCLUSION

In this paper, we have examined the role of the Cauchy integral in complex analysis and its connection with analytic and harmonic functions. Every analytic function has harmonic real and imaginary parts, a result that follows from the Cauchy–Riemann equations and the Cauchy integral formula. However, the converse is not true: a harmonic function need not be analytic unless additional conditions are satisfied. This highlights the fact that analyticity is a stronger and more restrictive property than harmonicity. The discussion presented here clarifies this distinction and reinforces the foundational importance of the Cauchy integral in complex analysis.

Competing Interests: The authors declare that they have no competing interests.

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