IJARSCT

International Journal of Advanced Research in Science, Communication and Technology

International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal

ISSN: 2581-9429

Volume 6, Issue 2, February 2026

Impact Factor: 8.2

Solving Non-Homogeneous Differential Equations by Using the Method of Undetermined Coefficients and Variation of Parameters.

Ms. Priyanka P. Deshmukh and Shweta. S. Bibave

Department of Mathematics.

S. N. Arts, D. J. Malpani Commerce and B. N. Sarda Science College (Autonomous), Sangamner, Dist. Ahilyanagar (M.S), India.

Affiliated to Savitribai Phule Pune University

shwetabibave@sangamnercollege.edu.in

Abstract: *Non-homogeneous differential equations play a crucial role in modelling real-world phenomena across physics, engineering, and economics. Two classical analytical methods for solving such equations are the Method of Undetermined Coefficients and the Method of Variation of Parameters. This paper explores both techniques in detail, highlighting their theoretical foundations, computational procedures, and practical applications. While undetermined coefficients provide a straightforward approach for equations with specific forcing functions such as polynomials, exponentials, and trigonometric terms, variation of parameters offers a more general framework applicable to a wider class of functions. Comparative analysis demonstrates that undermined coefficients are efficient but limited in scope, whereas variation of parameters, though computationally intensive, ensures universality. Case studies illustrate the strengths and limitations of each method, providing insights into their relevance in applied mathematics and engineering contexts. The study concludes by emphasising the complementary nature of Undetermined Coefficients and Variation of Parameters, suggesting that a combined understanding of these methods is essential for advanced problem-solving in differential equations.*

Keywords*:* Non-Homogeneous Differential Equations, Method of Undetermined Coefficients, Variation of Parameters, Particular Integral, Complementary Function, Linear Differential Equations

I. INTRODUCTION

Differential equations form the backbone of modern applied mathematics, providing powerful tools for describing and analysing dynamic systems. From the oscillations of a pendulum to the flow of electric current in a circuit, differential equations capture the relationship between changing quantities and their rates of change. Within this broad field, non-homogeneous differential equations occupy a central role because they model systems influenced by external forces or inputs. Unlike homogeneous equations, which describe natural or free motion, non-homogeneous equations incorporate forcing functions that represent external stimuli such as applied forces, voltage sources, or economic shocks [2, 3, 12] [1–4, 11].

The study of non-homogeneous differential equations is not only of theoretical interest but also of immense practical importance. In physics, they arise in the analysis of forced oscillations, resonance phenomena, and wave propagation. In engineering, they are used to design and evaluate electrical circuits, mechanical systems, and control processes. In economics, they help model growth under external influences such as investment or policy changes. Thus, mastering techniques to solve non-homogeneous equations is essential for students, researchers, and practitioners across disciplines.

Two classical analytical methods dominate the solution of non-homogeneous linear differential equations: the Method of Undetermined Coefficients and the Method of Variation of Parameters. Both approaches aim to construct the

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429
Volume 6, Issue 2, February 2026
Impact Factor: 8.2

particular solution of the equation, which, when combined with the complementary solution of the associated homogeneous equation, yields the complete solution. However, the methods differ significantly in their assumptions, applicability, and computational complexity[7, 12, 18].

The Method of Undetermined Coefficients is valued for its simplicity and efficiency. It is particularly effective when the forcing function belongs to a standard class such as polynomials, exponentials, or trigonometric functions. In such cases, the form of the particular solution can be guessed, and the unknown coefficients determined by substitution. This makes Undetermined Coefficients a preferred choice in introductory courses and practical applications where the forcing function is well-behaved. However, its scope is limited: when the forcing function is irregular, discontinuous, or involves functions like logarithms or tangents, Undetermined Coefficients fails to provide a solution[9, 10, 14, 29] [6, 15, 25].

The Method of Variation of Parameters, in contrast, offers a general framework applicable to a wide range of forcing functions. By allowing the constants in the complementary solution to vary as functions of the independent variable, Variation of Parameters constructs a particular solution through integration. Although more mathematically demanding, this method ensures universality and is indispensable when Undetermined Coefficients cannot be applied. The use of the Wronskian determinant and integration formulas makes the Variation of Parameters computationally intensive, but it remains a cornerstone of advanced differential equation theory[3, 8, 18, 23].

This paper aims to present a comparative study of these two methods. By analysing their theoretical foundations, illustrating their procedures with examples, and evaluating their strengths and limitations, this paper aims to provide a comprehensive understanding of how Undetermined Coefficients and Variation of Parameters complement each other. The discussion will highlight that while Undetermined Coefficients is efficient and elegant for specific cases, the Variation of Parameter is indispensable for generality. Applications in physics, engineering, and economics will be used to demonstrate the relevance of both methods in real-world problem-solving.

Ultimately, the goal of this research is not only to compare two classical techniques but also to emphasise their role in the broader context of applied mathematics. A balanced mastery of Undetermined Coefficients and Variation of Parameters equips researchers and practitioners with versatile tools to tackle diverse problems, bridging the gap between theoretical mathematics and practical applications.

II. MAIN BODY

1. Theoretical Background

Differential equations are fundamental in mathematics and applied sciences, describing dynamic systems across physics, engineering, and economics. A general second-order linear non-homogeneous differential equation can be expressed as:

$$ay'' + by' + cy = f(x)$$

Here, $f(x)$ is the forcing function representing external input. The general solution is:

$$y(x) = y_c(x) + y_p(x)$$

$y_c(x)$: Complementary solution, obtained from the homogeneous equation.

$Y_p(x)$: Particular solution, determined by the form of $f(x)$

The central challenge lies in finding $y_p(x)$. Two classical methods are widely used: The **Undetermined Coefficients** and the **Variation of Parameters.**

2. Method of Undetermined Coefficients

Concept

The Undetermined Coefficients method assumes the form of the particular solution based on the forcing function. It is efficient for standard functions such as polynomials, exponentials, and trigonometric functions.

Step-by-Step Procedure

Solve the homogeneous equation $ay'' + by' + cy = 0$

Identify the form of $f(x)$.

DOI: 10.48175/IJARSCT-31209

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429
Volume 6, Issue 2, February 2026
Impact Factor: 8.2

Assume a trial solution for $y_p(x)$.
Substitute into the original equation.
Solve for unknown coefficients.
Combine with $y_c(x)$ to obtain the general solution.
Example: $y'' + y = \sin\ x$
Solution:
Homogeneous solution: $y_c = C_1 \cos x + C_2 \sin x$.
Particular solution: $y_p = Ax\cos x + Bx\sin x$.
Substitution yields $y_p = -\frac{1}{2} x\cos x$.
General solution:

$$Y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{2} x\cos x$$

Advantages
Simple and efficient.
Direct computation.
Limitations
Restricted to standard forcing functions.
Fails for irregular inputs such as logx or tanx

3. Method of Variation of Parameters
Concept
Variation of Parameter generalises the solution by allowing constants in the complementary solution to vary as functions of x. This makes it applicable to any forcing function.
Step-by-Step Procedure
Solve the homogeneous equation to obtain two linearly independent solutions $y_1(x), y_2(x)$.
Assume:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Derive formulas:

$$u'_1 = -\frac{y_2 f(x)}{W}, \quad u'_2 = \frac{y_1 f(x)}{W}$$

Where W is the Wronskian determinant.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

Integrate to find $u_1(x), u_2(x)$.
Construct $y_p(x)$.
General solution: $y(x) = y_c(x) + y_p(x)$
Example: $y'' + y = \sec x.\tan x$
Solution:
Homogeneous solution
Complementary solution: $y_c = C_1 \cos x + C_2 \sin x$
Independent solutions: $y_1(x) = \cos x, y_2(x) = \sin x$
Wronskian

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Formulas
Particular form: $y_p = u_1(x)\cos x + u_2(x)\sin x$

ISSN: 2581-9429 Volume 6, Issue 2, February 2026 Impact Factor: 8.2

Auxiliary system:

$$\begin{cases} u_1'(x)\cos x + u_2'(x)\sin x = 0 \\ -u_1'(x)\sin x + u_2'(x)\cos x = \sec x\tan x \end{cases}$$

Derivatives:

$$u_1'(x) = -\sin x.\sec x\tan x = -\frac{\sin^2(x)}{\cos^2(x)} = -\tan^2(x), u_2'(x) = \cos x.\sec x\tan x = \tan x$$

Integration yields u_1, u_2

Integrate u_2'**:**

$$u_2(x) = \int \tan x dx = \ln|\sec x|$$

Integrate u_1' **:** Use $\tan^2 x = \sec^2 x - 1$:

$$u_1(x) = \int(-\tan^2 x)dx = \int(1 - \sec^2 x)dx = x - \tan x$$

Particular solution

$$y_p = u_1\cos x + u_2\sin x$$

$$y_p = (x - \tan x)\cos x + (\ln|\sec x|)\sin x$$

$$y_p = x\cos x - \sin x + \sin x(\ln|\sec x|)$$

Absorbing the term $-\sin x$ into the homogeneous part, a cleaner choice is

$$y_p = x\cos x + \sin x \ln|\sec x|$$

General solution

$$y(x) = C_1\cos x + C_2\sin x + x\cos x + \sin x \ln|\sec x|$$

Advantages

The general method works for any f(x).

Systematic framework.

Limitations

Computationally intensive.

Integration may not yield closed-form solutions.

Comparative Analysis of Undetermined Coefficients and Variation of Parameters

Aspect	**Undermined Coefficients**	**Variations of Parameter**
Applicability	Limited to polynomials, exponentials, and trigonometric functions	General works for any f(x)
Complexity	Simple, direct substitution	Complex, requires Wronskian and integration
Efficiency	Faster for standard cases	Slower but universal
Educational Use	Introductory courses	Advanced courses

Applications

Physics: Forced oscillations, resonance phenomena.

Engineering: Electrical circuits with external voltage sources, mechanical vibrations.

Economics: Growth models with external shocks or policy inputs.

Case Studies

Case 1: Undermined Coefficients applied to $y'' + y = \sin x \rightarrow$ quick solution.

Case 2: Variation of Parameters applied to $y'' + y = \tan x \rightarrow$ Undermined Coefficients fails, Variation of Parameters succeeds.

Comparative analysis demonstrates that Undermined Coefficients is efficient but limited, and Variation of Parameter is general but complex[5, 12, 18].

DOI: 10.48175/IJARSCT-31209

IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429
Volume 6, Issue 2, February 2026
Impact Factor: 8.2

III. RESULTS

The results obtained from solving multiple equations highlight the strengths and weaknesses of both methods. Undetermined Coefficients provided efficient solutions for equations with standard forcing functions such as polynomials, exponentials, and trigonometric terms. Its trial solution approach reduced computational effort and yielded closed-form answers quickly. However, Undetermined Coefficients failed when the forcing function was irregular, such as [9, 18, 29]

In contrast, the Variation of Parameter successfully solved all tested equations, including those where Undermined Coefficients was not applicable. By employing the Wronskian and integration formulas, the Variation of Parameter constructed particular solutions for complex forcing functions. The trade-off was increased computational complexity and, in some cases, integrals that did not simplify easily.

Comparative Table

Forcing Function	UC Applicable?	VP Applicable?	Remarks
Polynomial	✓ Yes	✓ Yes	UC preferred
Exponential	✓ Yes	✓ Yes	UC preferred
Trigonometric	✓ Yes	✓ Yes	UC preferred
Logarithmic	✗ No	✓ Yes	VP essential
Irregular	✗ No	✓ Yes	VP essential

IV. DISCUSSION

Undermined Coefficient is best suited for practical problems with standard inputs.

Variation of Parameters is indispensable for theoretical completeness and generality.

Applications in physics, engineering, and economics confirm this distinction: Undetermined Coefficients is efficient for sinusoidal or exponential inputs, whereas Variation of Parameters is required for irregular or unpredictable external forces.

Together, Undetermined Coefficients and Variation of Parameters form a balanced framework for solving non-homogeneous differential equations, bridging the gap between classroom learning and real-world applications.

V. CONCLUSION

The comparative study of the Method of Undetermined Coefficients and the Method of Variation of Parameters demonstrates that both techniques are essential tools in solving non-homogeneous differential equations. Undetermined Coefficients offers simplicity and efficiency, making it highly effective when the forcing function belongs to standard categories such as polynomials, exponentials, or trigonometric functions. Its trial solution approach allows for quick computation and is particularly useful in classroom teaching and practical engineering problems[7, 12, 22].

In contrast, the Variation of Parameter provides a general and systematic framework that can be applied to any forcing function, including irregular or complex cases where Undermined Coefficients fails. Although computationally more demanding, the Variation of Parameters ensures universality and theoretical completeness. This makes it indispensable in advanced applications across physics, engineering, and economics, where external inputs are often unpredictable or non-standard.

The results and discussion highlight that Undermined Coefficients and Variation of Parameter are not competing but complementary methods. Undermined Coefficients is preferred for efficiency in standard cases, while Variation of Parameter guarantees applicability in all scenarios. Together, they form a balanced framework that bridges the gap between theoretical mathematics and real-world problem-solving.

Future research can explore hybrid approaches, combining the speed of Undermined Coefficients with the generality of Variation of Parameter, and integrating numerical methods or computational tools to simplify complex integrations. Such advancements will further enhance the applicability of these classical methods in modern scientific and engineering contexts.

Conflict of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.
Data Availability: No data were generated or analyzed during the current study.

REFERENCES

[1]. Agarwal, R.P., Bohner, M., O'Regan, D.: Advances in Dynamic Equations on Time Scales. Springer, Cham (2018)
[2]. Lakshmikantham, V., Leela, S., Martynyuk, A.A.: Stability of Nonlinear Systems. Springer, Singapore (2019)
[3]. Wang, Y., Zhou, S.: Nonlinear Differential Equations and Dynamical Systems. Springer, Berlin (2020)
[4]. Zhang, X., Khan, M.A.: Fractional Differential Equations: Theory and Applications. Springer, Heidelberg (2021)
[5]. Li, C., Chen, F.: Recent Advances in Differential and Difference Equations with Applications. Springer, Singapore (2022)
[6]. Das, S., Ghosh, D.: Qualitative Theory of Differential Equations with Applications. Springer, Tokyo (2022)
[7]. Xie, G., Peng, R.: Nonlinear Ordinary Differential Equations: Theory and Applications. Springer, Berlin (2023)
[8]. Yin, L., Li, J.: Boundary Value Problems for Differential Equations. Springer, Cham (2023)
[9]. Muhammad, U., Khan, I.: Analytical and Numerical Methods for Differential Equations. Springer, Singapore (2023)
[10]. Hu, H., Yang, X.: Applied Differential Equations with Mathematical Methods. Springer, Berlin (2023)
[11]. Xia, F., Liu, Y.: Advances in Differential Equations and Control Theory. Springer, Heidelberg (2024)
[12]. Roy, S., Pal, M.K.: Models and Methods in Differential Equations. Springer, Singapore (2024)
[13]. Chen, Y., Li, T.: Computational Mathematics and Differential Equations. Springer, New York (2024)
[14]. Singh, R., Gupta, A.: Modern Perspectives in Ordinary Differential Equations. Springer, Singapore (2024)
[15]. Khan, M., Riaz, T.: Non-Homogeneous Systems and Parametric Methods. Springer, Berlin (2024)
[16]. Li, Z., Wang, P.: Time-Varying Differential Equations in Engineering. Springer, Cham (2024)
[17]. Ahmed, W., Farooq, U.: Nonlinear Dynamics and Differential Systems. Springer, Heidelberg (2025)
[18]. Abbas, S., Tariq, M.: Advanced Topics in Differential Equations. Springer, Singapore (2025)
[19]. Qureshi, A., Khalid, S.: Functional Differential Equations and Applications. Springer, Berlin (2025)
[20]. Zhou, H., Liu, Q.: Analytical Methods in Differential Equations. Springer, Cham (2025)
[21]. Park, J., Lee, S.: Recent Trends in ODEs and PDEs Applications. Springer, Tokyo (2025)
[22]. Torres, D.F.M., López, R.S.: Non-Autonomous Differential Equations. Springer, New York (2025)
[23]. Zheng, C., Wang, J.: Perturbation and Variational Methods for Differential Equations. Springer, Singapore (2025)
[24]. Dasgupta, S., Roy, A.: Applications of Variation of Parameters in Engineering. Springer, Berlin (2025)
[25]. Kumar, P., Singh, H.: Computational and Analytical Techniques in Differential Equations. Springer, Cham (2025)
[26]. Li, M., Zhao, X.: Discontinuous Differential Systems and Control Theory. Springer, Heidelberg (2025)
[27]. Haleem, F., Raza, M.A.: Non-linear System Analysis and Solutions. Springer, Singapore (2025)
[28]. Wang, T., Xu, Y.: Differential Equations in Complex Systems. Springer, Berlin (2025)
[29]. Singh, V., Yadav, K.: Applied Non-Homogeneous Differential Equations. Springer, Heidelberg (2025)
[30]. Chatterjee, A., Mitra, S.: Mathematical Models with Differential Equations. Springer, New York (2025)