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Suzuki Type Common Fixed Point Result on *h*-Metric Space

Jayashree Patil[1], Basel Hardan[2]* and Amol Bachhav[3]
Department of Mathematics, Vasantrao Naik Mahavidyalaya, Cidco, Aurangabad, India[1]
Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad India[2]
Navin Jindal School of Management, University of Texas at Dallas, Dallas, USA[3]
jv.patil29@gmail.com[1], bassil2003@gmail.com[2], amol.bachhav@utdallas.edu[3]

Abstract: *In this paper, we defined a general form of the type of Suzuki functions on h-metric space to obtain a common fixed point. Our results generalized some results from the literature.*

Keywords: Suzuki functions

I. Introduction

Banach [1] established a result which is known as Banach fixed point theorem or Banach contraction principle, (BCP) to prove the existence of solutions for integral equations and non-linear functional equations. Since then, BCP became a very common tool for solving a set of problems in different scientific fields. After that, a large quantity of literature is observed which applied, generalized, and expanded this results in several bearings by changing the hypothesizes, utilizing different setups. The need to build a suitable model that accurately measures the distance between two objects and more has prompted excited many authors, in which numbers of generalizations of metric spaces have shown in several papers, such as 2-metric spaces, n-metric spaces, partial metric spaces, and cone metric spaces. These generalizations were then utilized to expand the field of the study of fixed point results. More debate of these generalizations, we refer to [3, 4, 5, 12, 13, 15]. Boyd [2] expanded the BCP to the non-linear contraction mappings. We will start by recalling some fundamental results for h-metric spaces that will be needed in the complements. More specifics please see [6, 10].

Definition 1.1. [9] Let G be a non-empty set and $h : G^3 \to [0,1)$ be a function fulfilling the next conditions for all $x, y, z, w \in G$:
(hM1): $h(x, y, z) \geq 0$;
(hM2): $h(x, y, z) = 0$ if and only if $x = y = z$;
(hM3): $h(x, y, z) = h(x, x, z) + h(y, y, z) + h(z, z, w)$.
Then the function h is called an h-metric on G and the pair (G, h) is called an h-metric space

Example 1.2. [7] Suppose that $G = R^n$ and $\|\cdot\|$a norm on G , let $h(x, y, z) = \|y + z - 2x\| + \|y - z\|$, then$(G, h)$ is an h-metric space.

Definition 1.3. [8] Suppose that (G, h) be an h-metric space:
(i): A sequence$x_n \subset G$ converges to $x \in G$ if $h(x_n, x_n, x) \to 0$ as $n \to +\infty$, that is, for each $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$ we have$h(x_n, x_n, x) < \varepsilon$.
(ii): A sequence $y_n \subset G$ is called a Cauchy sequence if $(y_n, y_n, y_m) \to 0$as $n, m \to \infty$, that is, there exists $n_0 \in N$ such that for all $n, m \geq n_0$ we have $h(x_n, x_n, x_m) < \varepsilon$.
(iii): The h-metric space (G, h) is complete if every Cauchy sequence is a convergent sequence.

Definition 1.4. [7] Suppose that (G, h) be an h-metric space, for all radiuses$r > 0$ and center$x \in G$we consider the open ball $Bh(x; r) = \{y \in G : h(y, y, x) < r\}$

and closed ball $Bh[x;\ r] = \{y \in G : h(y, y, x) \leq r\}$.

Example 1.5. [7] Let $G = R\ and\ h(x, y, z) = |y + z - 2x| + |y - z|\ for\ all\ x, y, z \in G$, then

$$Bh(1,2) = \{y \in R : h(y, v, 1) < 2\}$$
$$= \{y \in R : |y - 1| < 1\}$$
$$= \{y \in R : 0 < y < 2\}$$
$$= (0, 2).$$

Lemma 1.6. [8] Let (G, h) be an h-metric space. If $r > 0$and $u \in G$, then the ball $Bh(x, r)$is open subset of G.

Lemma 1.7. [7, 8, 10] In an h-metric space, we have $h(x, x, y) = h(y, y, x)$.

Lemma 1.8. [10] Let (G, h) be an S-metric space. If sequence $|un|$ in converges to x, then x is unique.

Lemma 1.9. [10] Let (G, h) be an h-metric space. If sequence $\{x_n\} \subset G$ is converges to x, then fung is a Cauchy sequence.

Lemma 1.10. [7, 8, 10] Let (G, h)be an h-metric space. If there exist sequences $\{x_n\}, \{y_n\} \subset G$such that$\lim_{n\to\infty}\{x_n\} = x$ and $\lim_{n\to\infty}\{y_n\} = y$, then:

$$\lim_{n\to\infty} h(x_n, x_n, y_n) = h(x, x, y)$$

Definition 1.11. [7, 11] Let G be a non-empty set, a ϱ-metric on G is a function $\varsigma : G \times G \to [0; +\infty)$if there exists a real number $\varsigma \geq 1$ such that the following conditions hold for all $x, y, z \in G$,
(i): $\varsigma(x, y) = 0$, if and only if $x = y$,
(ii): $\varsigma(x, y) = \varsigma(y, x)$,
(iii): $\varsigma(x, z) = \varrho[\varsigma(x, y) + \varsigma(y, z)]$, the pair (G, ς) is called a ϱ-metric space.

Proposition 1.12. [8] Let (G, h) be an h-metric space and let $\varsigma(x, y) = h(x, x, y)$; for all $x, y \in G$. Then we have
(i): ς is a ϱ-metric on G,
(ii): $x_n \to x$ in (G, h) if and only if $x_n \to x$in (G, ϱ),
(iii): $\{x_n\}$ is a Cauchy sequence in (G, h) if and only if fung if a Cauchy sequence in (G, ϱ).

Definition 1.13. [14] Let φ be the set of all continuous functions $\vartheta : [0, 1)^4 \to [0, +\infty)$, satisfying the conditions:
(i): $\vartheta(1, 1, 1, 1) < 1$,
(ii): ϑ is sub homogeneous $\vartheta(\tau x_1, \tau x_2, \tau x_3, \tau x_4) < \vartheta\tau(x_1, x_2, x_3, x_4); \forall\tau \geq 0$,
(iii): if $x_i, y_i \in [0; +\infty)$; $x_i \leq y_i$for $i = 1,2,3,4$.We have

$$\vartheta(x_1, x_2, x_3, x_4) \leq \vartheta(y_1, y_2, y_3, y_4).$$

II. Main Results

Theorem 2.1. Suppose that (G, h) be an$h$-metric space and g, f are selfing mappings on G. Let there exist $\vartheta \in \varphi$and $\partial \in [0,1)$ such that $\partial(\theta + 2) \leq 1$ where, $\theta = \vartheta(1,1,1,1)$ and let that$\vartheta h(gx, gx, f(gx)) \leq h(gx, gy, gz)$ yields

$$h(f(gx), f(gy), f(gz)) \leq \vartheta\{h(gx, gy, gz), h(gx, gx, f(gx)), h(gy, gy, f(gy)),\ h(gz, gz, f(gz))\} \quad (2.1)$$

Then $F(f(gx))$ is non-empty set for all $x, y, z \in G$.

Proof. Let$f(gx_0) = gx_1$ for an arbitrary $x_0 \in G$. Since, $\partial h(gx_0, gx_0, f(gx_0)) < h(gx_0, gx_0, gx_1)$.
Thus, by inequality (2.1) and condition (iii) of Definition 1.13respectively, we obtain

$$h(gx_1, gx_1, f(gx_1)) = h(f(gx_0), f(gx_0), f(gx_1))$$
$$= \vartheta\{h(gx_0, gx_0, gx_1), h(gx_0, gx_0, gx_1), h(gx_0, gx_0, gx_1), h(gx_1, gx_1, f(gx_1))\}$$

Then, by Proposition 1.12, we get

$$h(gx_1, gx_1, f(gx_1)) \leq \theta h(gx_0, gx_0, gx_1).$$

Based on that we have

$$h(gx_2, gx_2, f(gx_2)) \leq \theta h(gx_1, gx_1, gx_1);$$

So, we have

$$h(gx_2, gx_2, gx_3) \leq \theta\, h(gx_1, gx_1, gx_2) \leq \theta^2\, h(gx_0, gx_0, gx_1).$$

Repeating this manner, we get a sequence $\{gx_n\}_{n\geq 1} \in G$, such that $f(gx_n) = gx_{n+1}$which fulfills:

$$h(gx_n, gx_n, gx_{n+1}) \leq \theta^n h(gx_0, gx_0, gx_1)(2.2)$$

Suppose that $gx_n \neq gx_{n+1}$ for each$n \geq 1$. Then for all $n < m$ by using $(h$M3),
Limma 1.7 and inequality (2.2), we have

$$h(gx_n, gx_n, gx_{n+m}) \leq 2h(gx_n, gx_n, gx_{n+1}) + h(gx_{n+m}, gx_{n+m}, gx_{n+1})$$
$$\leq 2h(gx_n, gx_n, gx_{n+1}) + 2h(gx_{n+1}, gx_{n+1}, gx_{n+2}) + 2h(gx_{n+m}; gx_{n+m}; gx_{n+2})$$
$$\leq 2\sum_{i=0}^{i=m-1} \theta^{i+n} h(gx_0, gx_0, gx_1) \leq h(gx_0, gx_0, gx_1)\frac{2\theta^n}{1-\theta}$$

By condition (i) of Definition 1.13and hypothesis of Theorem 2.1, we obtain

$$\lim_{n\to\infty} h(gx_n, gx_n, gx_{n+m}) \to 0$$

Therefore, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (G, h) and since (G, h) is a complete$h$-metric space. By Definition 1.3, then $\{x_n\}_{n\geq 1}$ conveges to some $x \in G$, i.e.,

$$\lim_{n\to\infty} h(gx_n, gx_n, x) = 0$$

Also,

$$\lim_{n\to\infty} h(gx_n, gx_{n+1}, x) = 0$$

Here, for all $n \geq 1$ we assume one of the following relations are holds

$$\partial h(gx_n, gx_n, f(gx_n)) \leq h(gx_n, gx_n, x);$$

or

$$\partial h(gx_{n+1}, gx_{n+1}, f(gx_{n+1})) \leq h(gx_n, gx_n, x):$$

By imposing the opposite, for some $n \geq 1$ we have:

$$\partial h(gx_n, gx_n, f(gx_n)) > h(gx_n, gx_n, x);$$
$$and$$
$$\partial h\big(gx_{n+1}, gx_{n+1}, f(gx_{n+1})\big) > h(gx_{n+1}, gx_{n+1}, x);$$
$$h(gx_n, gx_n, gx_{n+1}) \leq 2h(gx_n, gx_n, x) + h(gx_{n+1}, gx_{n+1}, x)$$
$$= \partial(2 + \theta)h(gx_n, gx_n, gx_{n+1}):$$

Then $\partial(2 + \theta) > 1$ but this contradicts the hypothesis of the theorem. Hence, our
assumption is proved. Note that from the assumption of the theorem, we have either

$$h\big(f(gx_n), f(gx_n), f(gx)\big) \leq \vartheta\left\{\begin{matrix} h(gx_n, gx_n, gx), h(f(gx_n), gx_n, gx); \\ h(f(gx_n), gx_n, gx), h(f(gx), gx_n, gx_n) \end{matrix}\right\}$$

Or,

$$h\big(f(gx_{n+1}), f(gx_{n+1}), f(gx)\big) \leq \vartheta\left\{\begin{matrix} h(gx_{n+1}, gx_{n+1}, gx), h(f(gx_{n+1}), gx_{n+1}, gx), \\ h(f(gx_{n+1}), gx_{n+1}, gx), h(f(gx), gx_{n+1}, gx_{n+1}) \end{matrix}\right\}$$

We have two cases:
Case (i): There exists an infinite subset $A \subset N$, such that for all $n \in A$

$$h(gx_{n+1}, gx_{n+1}, f(gx)) = h(f(gx_n), f(gx_n), f(gx))$$
$$\leq \vartheta\left\{\begin{matrix} h(gx_n, gx_n, gx), h(f(gx_n), gx_n, gx), \\ h(f(gx_n), gx_n, gx), h(f(gx), gx_n, gx_n) \end{matrix}\right\}$$
$$= \vartheta\left\{\begin{matrix} h(gx, gx, gx), h(f(gx), gx, gx), \\ h(f(gx), gx, gx), h(f(gx), gx, gx) \end{matrix}\right\}$$

Case (i): There exists an infinite subset$A \subset N$, such that for all $n \in A$

$$h(gx_{n+2}, gx_{n+2}, f(gu)) = h(f(gx_{n+1}), f(gx_{n+1}), f(gx))$$
$$\leq \vartheta\left\{\begin{matrix} h(gx_{n+1}, gx_{n+1}, gx), \quad h(f(gx_{n+1}), gx_{n+1}, gu); \\ h(f(gx_{n+1}), gx_{n+1}, gu), h(f(gu), gx_{n+1}, gx_{n+1}) \end{matrix}\right\}$$
$$= \vartheta\left\{\begin{matrix} h(gx, gx, gx), h(f(gx), gx, gx), \\ h(f(gx), gx, gx), h(f(gx), gx, gx) \end{matrix}\right\}$$

In case (i) and (ii), taking the limit as $n \to +\infty$ for all$n \in A$; and $n \in B$, such that $A \cap B = 0$ we obtain:

$$h(gx, gx, f(gx)) \leq \vartheta(0, 0, 0, h(f(gx), gx, gx),$$

using Definition 1.3and proposition 1.12, we get$h(gx, gx, f(gx)) = 0$, and Then $gx = f(gx))$. Hence, the proof is completed.

Corollary 2.2. Suppose that (G, h) be an$h$-metric space and g; f are selfing mappings on G. Let there exist $\vartheta \in \varphi$ and $\delta \in (0; 1]$, where$\theta = \vartheta(1, 1, 1, 1)$ and

$$h\big(f(gx), f(gy), f(gz)\big) \leq$$
$$\vartheta \max \delta\{h(gx;\ gy;\ gz), h\big(gx;\ gx;\ f(gx)\big), h\big(gy;\ gy;\ f(gy)\big), h\big(gz;\ gz;\ f(gz)\big)\}$$

Then g and f have a unique common fixed point.

Proof: Suppose that $\vartheta\ (gx_1;\ gx_2;\ gx_3;\ gx_4) = \delta \max\{gx_1;\ gx_2;\ gx_3;\ gx_4\}$.

References

[1]. S. Banach, Sur les operations dans les ensembles abstraits elleur application aux equations integrals, Fund. Math., 3 (1992), 133-181.

[2]. D. W. Boyd, S. W. Wong, On nonlinear contractions, Proc. Am. Math. Soc., 20 (1969), 458-464.

[3]. T. Do_senovi_c, S. Radenovic, S. Sedghi, Generalized metric spaces: Survey, TWMS. J. Pure Appl. Math., 9 (1) (2018), 3-17.

[4]. J. Esfahani, Z. D. Mitrovi_c, S. Radenovi_c, S. Sedghi, Suzuki-type point results in h-metric type spaces, Comm. Appl. Nonlinear Anal., 25 (3)(2018), 27{36.

[5]. S. Gahler, 2-metrische Raume und ihretopologischeStruktur, Math. Nachr., 26 (1963), 115-148.

[6]. N. Y. Ozgur, N. Tas, Some Fixed Point Theorems on S-metric Spaces, Mat. Vesnik, 69 (1) (2017), 39-52.

[7]. M. M. Rezaee, M. Shahraki, S. Sedghi, I. Altun, Fixed Point Theorems For Weakly ContractiveMappings On S-Metric Spaces And a Homotopy Result, Appl. Math. E-Notes, 17 (2017), 1607-2510.

[8]. S. Sedghi, N. V. Dung, Fixed Point Theorems on S-Metric Spaces, Mat. Vesnik, 66 (1) (2014), 113-124.

[9]. S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64 (2012), 258-266.

[10]. S. Sedghi, N. Shobe, T. Rako_cevi_c, Fixed Point Results In S-Metric Spaces, Nonlinear Funct. Anal. Appl., 20 (1) (2015), 55-67.

[11]. S. Sedghi, N. Shobe, M. Shahraki, T. Do_senovi_c, Common fixed Point of four maps in S-metric Spaces, Math. Sci., 12 (2018), 137-143.

[12]. S. Sedghi, A. Gholidahneh, T. Do_senovi_c, J. Esfahani, S. Radenovi_c, Common fixed point of four maps in b-metric spaces, J. Linear Topol. Algebra, 05 (02) (2016), 93-104.

[13]. S. Sedghi, M. M. Rezaee, T. Do_senovific, S. Radenovific, Common fixed point theorems for contractive mappings satisfying k-maps in h-metric spaces, Acta Univ. Sapientiae, Mathematica, 8 (2) (2016), 298-311.

[14]. M. Shahraki, S. Sedghi, S. M. A. Aleomraninejad, Zoran D. Mitrovific, Somefixed point results on S-metric spaces, Acta Univ. Sapientiae, Mathematica, 12, 2 (2020) 347-357.

[15]. V. Todorficevific, Harmonic Quasi conformal Mappings and Hyperbolic Type Metrics, Springer Nature Switzerland AG 2019.