IJARSCT

ISSN (Online) 2581-9429

Impact Factor: 6.252

International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)

Volume 2, Issue 3, March 2022

Generalized Monotone Method for Caputo Fractional Reaction-Diffusion Equation

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Abstract: *In this paper, our aim is to obtain the integral representation for the solution of non-linear Caputo reaction-diffusion equation of order q, where 0 < q < 1, in term of Green's function. We have developed a generalized monotone method for non-linear weakly coupled Caputo reaction-diffusion equation. The generalized monotone method yields monotone sequences which converges uniformly and monotonically to coupled minimal and maximal solutions. The existence of a unique solution for the non-linear Caputo reaction-diffusion equation is obtained.*

Keywords: Caputo Fractional Derivative; Eigen Function; Non-Linear Weakly Coupled System; Coupled Upper And Lower Solutions; Generalized Monotone Method

I. Introduction

Computation of explicit solution of non-linear dynamic equation is rarely possible. It is more so with non-linear fractional dynamic equations with initial and boundary conditions. In general, the existence and uniqueness of solution of the fractional dynamic equation has been established mostly, using some kind of fixed point approach. See [4, 5, 13, 10, 22, 21, 2, 25]. The method of upper and lower solutions combined with the monotone iterative technique not only guarantees the interval of existence but also the method is both theoretical and computational [7, 8, 9, 26]. The idea is to construct a sequence of approximate solutions which are either monotonically increasing or monotonically decreasing, if the approximation is the lower solution or upper solution respectively. In order to handle such problems, a generalized monotone method has been developed in [24, 14, 15, 16, 17].

In this paper, we consider the non-linear Caputo reaction-diffusion equation we develop generalized monotone method for the non-linear weakly coupled Caputo reaction-diffusion equation using coupled lower and upper solutions. Initially, we ob- tain a representation form for the solution of linear weakly coupled Caputo reaction- diffusion equation using the eigen function expansion method and Green's identity. These results are used to prove the sequences developed in the generalized monotone method converge to the coupled minimal and maximal solutions of the non-linear fractional diffusion equations. The convergence of the sequences is monotonic and uniform in the weighted norm.

The rest of paper is arranged in the following way. In section 2, definitions and basic results are discussed that plays vital role in the main results. In section 3, comparison results are obtained. These results are used to obtain section 4 deals to develops main results generalized monotone method converging to coupled minimal and maximal solutions of the non-linear Caputo fractional reaction-diffusion equation. Finally we prove that there exists a unique solution to the non-linear Caputo reaction- diffusion equation.

II. Definitions and Basic Results

In this section, we recall some definitions and results which are useful to develop our main result.

Definition 2.1 The Gamma function $\Gamma(q)$, is defined by

$$\Gamma(q) = \int_0^\infty s^{q-1} e^{-s} ds. \tag{2.1}$$

Definition 2.2 The Caputo (left-sided) fractional derivative of u(t) of order q when $1 < q \leq n$, is defined as

$$^{c}D^{q}u(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} u^n(s) ds, \quad t \in [0, \infty), \quad t > t_0 \qquad (2.2)$$

In particular, if q = n, an integer, then $^{c}D^{q}u = u^{n}(x)$ and $^{c}D^{q}u = u'(x)$ if q = 1.

Definition 2.3 The Riemann-Liouville fractional Integral of u(t) of order q when $o < q \leq 1$, is defined as

$$D^{-q}u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds, \qquad (2.3)$$

Definition 2.4 The Riemann-Liouville (left-sided) fractional derivative of u(t) of order q when $o < q < 1$, is defined as

$$D^{q}u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{q-1} u(s) ds, t > 0 \qquad (2.4)$$

Note that the Caputo integral of order q for any function is same as the Riemann- Liouville integral. Definition 2.5 The two parameter Mittag-liffler function is defined as

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+r)} \qquad (2.5)$$

For more details, see [1, 23, 19] In our next definition we assume p = 1 −q. When 0 < q < 1, J = (0, T] and J0 = [0, T].

Definition 2.6 A function φ(t) ∊ C(J, R) is a Cp continuous function, if t^{1-q}φ(t) ∊ C(J0, R). The set of Cp continuous functions is denoted by Cp(J, R) Further, given a function φ(t) Cp(J, R), we call the function t1−qφ(t) the continuous extension of φ(t).
Note that any continuous function in J0 is also a Cp continuous function.
Consider the initial value problem for the linear Caputo fractional differential equation of order q as

$$^{c}D^{q}u = \lambda u + f(t), \Gamma(q)u(t)t^{1-q}|_{t=0} = u^0, \qquad (2.6)$$

where λ is a real number and f ∊ C[J0, R]. The integral representation of the solution of equation (2.6) is:

$$u(t) = u^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}[\lambda(t-s)^q] f(s) ds. \qquad (2.7)$$

For details, see [3]. The next result is a basic comparison result involving the qth order fractional Caputo derivative with respect to time.

Lemma 2.1 [7, 8]. Let m(x, t) ∈Cp[J0, R] be such that for some t1 ∈(0, T], m(x, t1) = 0, and t1−qm(x, t) ≤ 0 on [0, t1], then Dqm(x, t1) ≥ 0.

III. Auxiliary Results

In this section, we obtain a representation form for the solution of the non-linear Caputo fractional reaction-diffusion equation with the fractional time derivative we also prove using the eigen function expansion method. We develop comparison results for the non-linear Caputo fractional reaction-diffusion equation with initial and boundary conditions. The comparison theorem is with respect to the lower and upper solutions when the non-linear term is of the form f (x, t, u) where f (x, t, u) satisfies one sided Lipschitz condition. In this case, we assume the non-linear in u for (x, t)in [0, L] [0, T]. In order to present our result, consider the non-linear Caputo fractional diffusion equation with initial and boundary conditions of the form

$$^{c}\partial_t^q u - k u_{xx} = f(x,t,u) \quad on \quad Q_T. \tag{3.1}$$

$$u(0,t) = A(t), u(L,t) = B(t) \quad in \quad \Gamma_T.$$

$$\Gamma(q)t^{1-q}u(x,t)|_{t=0} = f^0(x) \quad x \in \Omega$$

where Ω = [0, L], J = (0, T], QT = J × Ω, k > 0 and ΓT = (0, T) × ∂Ω. Here $\frac{\partial^q u}{\partial t}$ is the partial Caputo fractional derivative with respect to time 't'of order q, 0 < q < 1. In order for the initial boundary value problem to be compatible, we assume that f 0(0) = A(0) = f 0(L) = B(0) = 0, $\Gamma(q)t^{1-q}$u(x, t) t=0 = f 0(x). Here and throughout this work, we assume the initial and boundary condition satisfy the compatibility conditions. Using the method of eigen function expansion the solution of (3.1) of the form:

$$u(x,t) = \sum_{k=0}^{\infty} b_n(t)\phi_n(x), \tag{3.2}$$

where the eigenfunctions of the related homogeneous problem are known to be φn(x) = sinnπx/L and its corresponding eigenvalues are λn = $(n\pi/L)^2$. Using the same approach as in [6, 20, 12]. We can compute $b_n(t)$, where $b_n(t)$ will be the solution of the ordinary linear Caputo differential equation.

Using the standard arguments, one can compute bn(t) as follows.

$$b_n(t) = b_n^0 t^{q-1}E_{q,q}(-k\lambda_n t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)q_n(s) + k\frac{2n\pi}{L^2}[A(s)-(-1)^n B(s)]ds \tag{3.3}$$

where

$$b_n^0 = \frac{2}{L}\int_0^L f^0(y)\phi_n(y)dy \tag{3.4}$$

$$q_n(t) = \frac{2}{L}\int_0^L f(x,t,u)\phi_n(y)dy. \tag{3.5}$$

Therefore,

$$\begin{aligned} b_n(t) =& \frac{2}{L}\int_0^L f^0(y)\phi_n(y)dy t^{q-1}E_{q,q}(-k\lambda_n t^q) \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)\frac{2}{L}\int_0^L f(x,t,u)\phi_n(y)dyds \\ &+ k\frac{2n\pi}{L^2}\int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)[A(s)-(-1)^n B(s)]ds. \end{aligned}$$

So, using $b_n(t)$ in (3.2), we can get the solution $u(x,t)$ has the form

$$\begin{aligned} u(x,t) =& \int_0^L t^{q-1}[\sum_{k=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n t^q)\phi_n(x)\phi_n(y)]f^0(y)dy \\ &+ \int_0^t\int_0^L[\sum_{k=1}^{\infty}\frac{2}{L}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]f(x,t,u)dyds \\ &+ k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]A(s)ds \\ &- k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]B(s)ds. \end{aligned}$$

IJARSCT
ISSN (Online) 2581-9429
International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)
Impact Factor: 6.252
Volume 2, Issue 3, March 2022

Finally, we can write

$$u(x,t) = \int_0^L t^{q-1} G(x,y,t) f^0(y) dy + \int_0^t \int_0^L G(x,y,t-s) f(x,t,u) dy ds$$
$$+ k \int_0^t G_y(x,0,t-s) A(s) ds - k \int_0^t G_y(x,L,t-s) B(s) ds,$$

Where

$$G(x,y,t) = \sum_{k=0}^{\infty} \frac{2}{L} E_{q,q}(-k\lambda_n t^q) \phi_n(x) \phi_n(y).$$

This result is useful in our main result for computing the linear approximations of the generalized monotone iterates. We recall lemmas regarding the Mittage-Leffler function series from.

Lemma 3.1[4] Let $E_{q,1}(-\lambda t^q)$ be the Mittage-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.

Lemma 3.2[4] Let $E_{q,q}(-\lambda t^q)$ be the Mittage-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.

Now, we show the convergence of the above solution using Lemma 3.1 and Lemma 3.2 above. We can split the solution of (3.1)as u1(x, t), u2(x, t) and u3(x, t) respectively as follows:
(a) $u_1(x, t)$ is the solution of (3.1), when $f(x, t, u) = 0$, $A(t) = 0 = B(t)$,
(b) $u_2(x, t)$ is the solution of (3.1), when $A(t) = 0 = B(t)$, $f^0 = 0$,
(c) $u_3(x, t)$ is the solution of (3.1), when $f(x, t, u) = 0$, $f^0 = 0$.

Theorem 3.1[4] $u_1(x, t)$, $u_2(x, t)$ and $u_3(x, t)$ converge when $|f^0 = 0| < N_1$, $N_1 > 0$, $f(x, t, u)| < N_2$, $N_2 > 0$; $|A(t)| < M_1$, $M_1 > 0$ and $|B(t)| < M_2$, $M_1, M_2 > 0$ respectively.
Now, we consider the weekly coupled of non-linear Caputo fractional reaction diffusion equations of the type:

$$^c\partial_t^q u(x,t) - k\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t,u(x,t)) \quad on \quad Q_T. \tag{3.6}$$

$$\Gamma(q) t^{1-q} u(x,t)|_{t=0} = f^0(x), \quad x \in \overline{\Omega},$$
$$u(0,t) = A(t), u(L,t) = B(t) \quad on \quad \Gamma_T,$$
$$\Omega = [0,L], J = (0,T], Q_T = J \times \Omega, k > 0,$$
$$\Gamma_T = (0,T) \times \partial\Omega$$
$$f \quad in C^{2,q}[\Omega \times J \times \mathbb{R}, \mathbb{R}].$$

In this work, we seek the classical solution of (3.6) $u(x,t) \in C_p^{2,q}$ on Q_T, and $u(x,t) \in C_p$ on $\overline{Q_T}$. To develop the generalized monotone method for (3.6), non-linear of Caputo reaction-diffusion equation we need to define function.

Definition 3.1 If the functions $v(x, t), w(x, t) \in C^{2,q}[Q_T, R]$ are called lower and upper solutions of (3.6) if

$$^c\partial_t^q v(x,t) - k\frac{\partial^2 v(x,t)}{\partial x^2} \leq f(x,t,v(x,t)) \quad on \quad Q_T \tag{3.7}$$

$$\Gamma(q)(t-t_0)^{1-q}v(x,t)|_{t=0} \leq f^0(x), \quad x \in \overline{\Omega}$$
$$v(x,0) \leq A(t), v(L,t) \leq B(t) \quad in \quad \Gamma_T,$$

and

$$^c\partial_t^q w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} \geq f(x,t,w(x,t)), \quad on \quad Q_T \tag{3.8}$$
$$\Gamma(q)(t-t_0)^{1-q}w(x,t)|_{t=0} \geq f^0(x), \quad x \in \overline{\Omega}$$
$$w(x,0) \geq A(t), w(L,t) \geq B(t) \quad in \quad \Gamma_T.$$

Definition 3.2 A function f (x, t, u) in $C^{2,q}[\Omega \times J \times R, R]$ is said to be quasi- monotone nondecreasing if f (x, t, u) ≤ f (x, t, v) if u ≤ v.

Definition 3.3 A function f (x, t, u in $C^{2,q}[\Omega \times J \times R, R]$ is said to be quasi- monotone nonincreasing if f (x, t, u) ≥ f (x, t, v) if u ≤ v.

The next result is a comparison result relative to lower and upper solutions of (3.6).

Theorem 3.2 Assume that

(i) $v(x,t), w(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are natural lower and upper solutions of (3.6), respectively and $\Gamma(q)t^{1-q}v(x,t)|_{t=0} \leq \Gamma(q)t^{1-q}w(x,t)|_{t=0}$, $v(0,t) \leq w(0,t)$, $v(L,t) \leq w(L,t)$.

(ii) $f(x,t,u)$ satisfies the one sided Lipschitz condition

$$f(x,t,u_1) - f(x,t,u_2) \leq M(u_1 - u_2),$$

whenever $u_1 \geq u_2$ and $L > 0$. Then $v(x,t) \leq w(x,t)$ on $J \times \Omega$.

Proof. Initially, we prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m(x,t) = v(x,t) - w(x,t)$. We claim that $m(x,t) < 0$, $(x,t) \in \Omega \times J$. Suppose that the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{q-1}m(x_1,t_1) < 0$ on $[0,t_1)$, $m(x_1,t_1) = 0$. It easy to check $m_x(x_1,t_1) = 0$ and $\frac{\partial^2 m(x_1,t_1)}{\partial x^2} \leq 0$.
Then, using Lemma 2.1 we get $^c\partial_t^q m(x_1,t_1) \geq 0$
From the hypothesis, we also have

$$\begin{aligned}
&^c\partial_t^q m(x_1,t_1) \\
&\quad = {}^c\partial_t^q v(x_1,t_1) - {}^c\partial_t^q w(x_1,t_1) \\
&\quad < k\frac{\partial^2 v(x_1,t_1)}{\partial x^2} + f(x_1,t_1,v(x_1,t_1)) - k\frac{\partial^2 w(x_1,t_1)}{\partial x^2} -_f (x_1,t_1,w(x_1,t_1)) \\
&\quad < f(x_1,t_1,v(x_1,t_1)) - f(x_1,t_1,w(x_1,t_1)) = 0
\end{aligned}$$

which is a contradiction. Therefore, $v(x,t) < w(x,t)$ on $\overline{Q_T}$.
In order to prove the theorem for the non strict inequalities, let

$$\overline{w}(x,t) = w(x,t) + \epsilon t^{q-1}E_{q,q}[2Mt^q],$$
$$\overline{v}(x,t) = v(x,t) - \epsilon t^{q-1}E_{q,q}[2Mt^q].$$

IJARSCT

ISSN (Online) 2581-9429

International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)

Volume 2, Issue 3, March 2022

Impact Factor: 6.252

From this it follows

$$\overline{w}(0,t) > \overline{v}(0,t),$$
$$\overline{w}(L,t) > \overline{v}(L,t),$$
$$\Gamma(q)t^{1-q}\overline{w}(x,t)|_{t=0} > \Gamma(q)t^{1-q}w(x,t)|_{t=0} > \Gamma(q)t^{1-q}v(x,t)|_{t=0} > \Gamma(q)t^{1-q}\overline{v}(x,t)|_{t=0}.$$

Then,

$$\begin{aligned}
{}^{c}\partial_t^q\overline{w}(x_1,t_1) - k\frac{\partial^2\overline{w}(x,t)}{\partial x^2}
&= {}^{c}\partial_t^q\overline{w}(x_1,t_1) - k\frac{\partial^2 w(x,t)}{\partial x^2} + {}^{c}\partial_t^q \epsilon t^{q-1}E_{q,q}[2Mt^q] \\
&\geq f(x,t,w(x,t)) + \epsilon t^{q-1}E_{q,q}2ME_{q,q}[2Mt^q] \\
&= f(x,t,w(x,t)) + \epsilon M t^{q-1}E_{q,q}[2Mt^q] - f(x,t,\overline{w}(x,t)) + f(x,t,\overline{w}(x,t)) \\
&\geq -M(\overline{w} - w) + f(x,t,\overline{w}(x,t) + \epsilon 2ME_{q,q}[2Mt^q] \\
&= -M\epsilon t^{q-1}E_{q,q}[2Mt^q] + f(x,t,\overline{w}(x,t) + \epsilon 2ME_{q,q}[2Mt^q] \\
&= f(x,t,\overline{w}(x,t) + \epsilon ME_{q,q}[2Mt^q] \\
&> f(x,t,\overline{w}(x,t)) \quad on \quad \overline{Q_T}.
\end{aligned}$$

Similarly,

$${}^{c}\partial_t^q\overline{v}(x_1,t_1) - k\frac{\partial^2\overline{v}(x,t)}{\partial x^2} > f(x,t,\overline{v}(x,t)) \quad on \quad \overline{Q_T}$$

By the strict inequality result, $\overline{v} < \overline{w}$ on $\overline{Q_T}$. Letting $\epsilon \to 0$ we have $v \leq w$ on $\overline{Q_T}$.

The next result is the maximum principle for the Caputo parabolic equation in one dimensional space which will be useful in proving the uniqueness of the solution.

Corollary 3.1 Let

$$\begin{aligned}
{}^{c}\partial_t^q m(x_1,t_1) - k\frac{\partial^2 m(x,t)}{\partial x^2} &\leq 0 \quad on \quad Q_T, \\
m(0,t) \leq 0, m(L,t) &\leq 0 \quad on \quad \Gamma_T, \\
\Gamma(q)t^{1-q}m(x,t)|_{t=0} &\leq 0 \quad on \quad \overline{\Omega}.
\end{aligned}$$

Then $m(x,t) \leq 0$ on Q_T.

Proof. Suppose $m(x,t)$ has positive maximum at (x_1,t_1). Let $m(x_1,t_1) = K$. Let $\overline{m}(x,t) = m(x,t) - K$. Then $t^{q-1}\overline{m}(x,t) \leq 0$ on $(0,t_1]$ and $\overline{m}(x_1,t_1) = 0$. Using lemma (2.1) we get ${}^{c}\partial_t^q m(x_1,t_1) \geq 0$. Also $\frac{\partial^2 m(x_1,t_1)}{\partial x^2} \leq 0$. Combining these two, we get ${}^{c}\partial_t^q m(x_1,t_1) - k\frac{\partial^2 m(x_1,t_1)}{\partial x^2} \geq 0$.
Also, we have

$${}^{c}\partial_t^q m(x,t) - K\frac{\partial^2\overline{m}(x,t)}{\partial x^2} = {}^{c}\partial_t^q m(x,t) - K\frac{\partial^2 m(x,t)}{\partial x^2} - K\frac{t^{q-1}}{\Gamma q} < {}^{c}\partial_t^q m(x,t) - k\frac{\partial^2 m(x,t)}{\partial x^2} < 0 \tag{3.9}$$

which gives a contradiction. Hence, $m(x,t) \leq 0$.

IJARSCT
ISSN (Online) 2581-9429
International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)
Volume 2, Issue 3, March 2022
Impact Factor: 6.252

The solution of the linear problem is unique which follows from this maximum principle. This maximum principle is used to show the uniqueness of iterates and monotonicity of this iterates.

IV. Main Results

In this section, we develop a generalized monotone method for the nonlinear Caputo fractional reaction-diffusion equation (3.6) using coupled lower and upper solution. The generalized monotone method yields monotone sequences which converge uni- formly and monotonically to coupled minimal and maximal solutions of (3.6). We prove the uniqueness of the solution of (3.6).

Theorem 4.1 (i) Let $f(x,t,u)$ in $C^{2,q}[\Omega \times J \times \mathbb{R}^2, \mathbb{R}]$ be quasimonotone nondecreasing.
(ii) Let (v^0, w^0) be the coupled lower and upper solutions of (3.6) such that $t^{1-q}v^0 \leq t^{1-q}w^0$ on $\overline{Q_T}$.
(iii) Let $f(x,t,u)$ satisfies the one sided Lipschitz condition

$$f(x,t,u_1) - f(x,t,u_2) \geq -M(u_1 - u_2),$$

whenever $u_2 \leq u_1$ and $M > 0$. Then there exist monotone sequences $\{t^{1-q}v^n(x,t)\}$ and $\{t^{1-q}w^n(x,t)\}$ such that $t^{1-q}v^n(x,t) \to t^{1-q}\rho(x,t)$ and $t^{1-q}w^n(x,t) \to t^{1-q}\gamma(x,t)$ uniformly and monotonically on $\overline{Q_T}$, where $\rho(x,t)$ and $\gamma(x,t)$ are coupled minimal and maximal solutions of (3.6) respectively.

Proof. We construct the sequences $\{v^n(x,t)\}$ and $\{w^n(x,t)\}$ as follows:

$$ {}^c\partial_t^q v^n(x,t) - k\frac{\partial^2 v^n(x,t)}{\partial x^2} = f(x,t,v^{n-1}(x,t)), \quad on \quad Q_T \tag{4.1}$$

$$\Gamma(q)(t)^{1-q}v^n(x,t)|_{t=0} = f^0(x),$$
$$v^n(x,0) = A(t), v^n(L,t) = B(t).$$

and

$$ {}^c\partial_t^q w^n(x,t) - k\frac{\partial^2 w^n(x,t)}{\partial x^2} = f(x,t,w^{n-1}(x,t)), \quad on \quad Q_T \tag{4.2}$$

$$\Gamma(q)(t)^{1-q}w^n(x,t)|_{t=0} = f^0(x),$$
$$w^n(x,0) = A(t), w^n(L,t) = B(t).$$

It is easy to observe that $v^1(x,t)$ and $w^1(x,t)$ exist and unique by the representation form of linear equation and Corollary 3.1. By induction and the assumptions on $f(x,t,u(x,t))$ we prove that the solution $v^n(x,t)$ and $w^n(x,t)$ exist and unique by Corollary 3.1, for any n. Let us prove first $v^0(x,t) \leq v^1(x,t)$ and $w^1(x,t) \leq w^0(x,t)$ on Q_T. Let $\rho(x,t) = v^0(x,t) - v^1(x,t)$. Then

$$\begin{aligned} {}^c\partial_t^q \rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2} &= {}^c\partial_t^q v^o(x,t) - k\frac{\partial^2 v^0(x,t)}{\partial x^2} - \left[{}^c\partial_t^q v^1(x,t) - k\frac{\partial^2 v^1(x,t)}{\partial x^2}\right] \\ &\leq f(x,t,v^0(x,t)) - [f(x,t,v^0(x,t))] = 0 \end{aligned}$$

DOI: 10.48175/IJARSCT-3088

$\rho(0,t)=0, \rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho(x,t)\leq 0$ on $\overline{Q_T}$ and $t^{1-q}v^0(x,t)\leq t^{1-q}v^1(x,t)$ on $\overline{Q_T}$. Assume that $v^{k-1}(x,t)\leq v^k(x,t)$. Now we show $v^k(x,t)\leq v^{k+1}(x,t)$. Let $\rho(x,t)=v^k(x,t)-v^{k+1}(x,t)$. Then

$$\begin{aligned}{}^c\partial_t^q\rho(x,t)-k\frac{\partial^2\rho(x,t)}{\partial x^2} &= {}^c\partial_t^q v^k(x,t)-k\frac{\partial^2 v^k(x,t)}{\partial x^2}-\left[{}^c\partial_t^q v^{k+1}(x,t)-k\frac{\partial^2 v^{k+1}(x,t)}{\partial x^2}\right]\\ &\leq f(x,t,v^{k-1}(x,t),)-[f(x,t,v^k(x,t))]\\ &\leq M(v^{k-1}(x,t)-v^k(x,t))\\ &\leq M\rho(x,t)\end{aligned}$$

$\rho(0,t)=0, \rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho(x,t)\leq 0$ on $\overline{Q_T}$ and $t^{1-q}v^k(x,t)\leq t^{1-q}v^{k+1}(x,t)$ on $\overline{Q_T}$. Hence by mathematical induction, we have

$$t^{1-q}v^0(x,t)\leq t^{1-q}v^1(x,t)...t^{1-q}v^k(x,t)\leq t^{1-q}v^{k+1}(x,t)...t^{1-q}v^{n-1}(x,t)\leq t^{1-q}v^n(x,t) \tag{4.3}$$

We show that $w^1(x,t)\leq w^0(x,t)$ on $\overline{Q_T}$.
Let $\rho(x,t)=w^1(x,t)-w^0(x,t)$. Then

$$\begin{aligned}{}^c\partial_t^q\rho(x,t)-k\frac{\partial^2\rho(x,t)}{\partial x^2} &= {}^c\partial_t^q w^1(x,t)-k\frac{\partial^2 w^1(x,t)}{\partial x^2}-\left[{}^c\partial_t^q w^0(x,t)-k\frac{\partial^2 w^0(x,t)}{\partial x^2}\right]\\ &\leq f(x,t,w^0(x,t))-[f(x,t,w^0(x,t))]=0\end{aligned}$$

$\rho(0,t)=0, \rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho(x,t)\leq 0$ on $\overline{Q_T}$ and $t^{1-q}w^0(x,t)\leq t^{1-q}w^1(x,t)$ on $\overline{Q_T}$. Assume that $w^k(x,t)\leq w^{k-1}(x,t)$. To show that $w^{k+1}(x,t)\leq w^k(x,t)$.
Let $\rho(x,t)=w^{k+1}(x,t)-w^k(x,t)$. Then

$$\begin{aligned}{}^c\partial_t^q\rho(x,t)-k\frac{\partial^2\rho(x,t)}{\partial x^2} &= {}^c\partial_t^q w^{k+1}(x,t)-k_i\frac{\partial^2 w^{k+1}(x,t)}{\partial x^2}-\left[{}^c\partial_t^q w^k(x,t)-k\frac{\partial^2 w^k(x,t)}{\partial x^2}\right]\\ &\leq f(x,t,w^k(x,t))-[f_i(x,t,w^{k+1})]\\ &\leq M(w^k(x,t)-w^{k+1}(x,t))\\ &\leq -M\rho(x,t)\end{aligned}$$

$\rho(0,t)=0, \rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho(x,t)\leq 0$ on $\overline{Q_T}$ and $t^{1-q}w^{k+1}(x,t)\leq t^{1-q}v^k(x,t)$ on $\overline{Q_T}$. Hence by mathematical induction, we have

$$t^{1-q}w^n(x,t)\leq t^{1-q}w^{n-1}(x,t)...t^{1-q}w^{k+1}(x,t)\leq t^{1-q}w^k(x,t)...t^{1-q}w^1(x,t)\leq t^{1-q}w^0(x,t) \tag{4.4}$$

IJARSCT

ISSN (Online) 2581-9429

International Journal of Advanced Research in Science, Communication and Technology (IJARSCT)

Volume 2, Issue 3, March 2022

Impact Factor: 6.252

Then, we prove that $v^1(x,t) \leq w^1(x,t)$. Let $\rho(x,t) = v^1(x,t) - w^1(x,t)$. Then from hypothesis, we get

$$\begin{aligned} {}^c\partial_t^q \rho(x,t) - k\frac{\partial^2 \rho_i(x,t)}{\partial x^2} &= {}^c\partial_t^q v^1(x,t) - k\frac{\partial^2 v^1(x,t)}{\partial x^2} - \left[{}^c\partial_t^q w^1(x,t) - k\frac{\partial^2 w^1(x,t)}{\partial x^2}\right] \\ &\leq f(x,t,v^0(x,t)) - [f(x,t,w^0(x,t))] \\ &\leq M(v^0(x,t) - w^0(x,t)) \\ &\leq -M\rho(x,t) \end{aligned}$$

$\rho(0,t) = 0, \rho(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}\rho(x,t)\mid_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho(x,t) \leq 0$ on $\overline{Q_T}$ and $t^{1-q}v^1(x,t) \leq t^{1-q}w^1(x,t)$ on $\overline{Q_T}$.
Hence,
$t^{1-q}v^0(x,t) \leq t^{1-q}v^1(x,t) \leq t^{1-q}w^1(x,t) \leq t^{1-q}w^0(x,t)$ on $\overline{Q_T}$.
By mathematical induction and equations (4.3), (4.4) we have
$t^{1-q}v^0(x,t) \leq ... \leq t^{1-q}v^n(x,t) \leq t^{1-q}w^n(x,t) \leq ... \leq t^{1-q}w^0(x,t)$ on $\overline{Q_T}$ for all n.
Furthermore, if $t^{1-q}v^0(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w^0(x,t)$ on $\overline{Q_T}$, then for any $u(x,t)$ of (3.6), we establish the following inequality by the method of induction.

$$t^{1-q}v^0(x,t) \leq ... \leq t^{1-q}v^n(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w^n(x,t) \leq ... \leq t^{1-q}w^0(x,t) \tag{4.5}$$

on $\overline{Q_T}$ for all n.
It is certainly true for $n = 0$, by hypothesis. Assume the inequality (4.3) to be true for $n = k$, that is

$$t^{1-q}v^0(x,t) \leq ... \leq t^{1-q}v^k(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w^k(x,t) \leq ... \leq t^{1-q}w^0(x,t) \tag{4.6}$$

on $\overline{Q_T}$ for all n.
Let$\rho(x,t) = v^{k+1}(x,t) - u(x,t)$. Then from hypothesis, we get

$$\begin{aligned} {}^c\partial_t^q \rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2} &= {}^c\partial_t^q v^{k+1}(x,t) - k\frac{\partial^2 v^{k+1}(x,t)}{\partial x^2} - \left[{}^c\partial_t^q u(x,t) - k\frac{\partial^2 u(x,t)}{\partial x^2}\right] \\ &\geq f(x,t,v^{k+1}(x,t)) - [f_i(x,t,u(x,t))] \\ &\geq -M(v^{k+1}(x,t) - u(x,t)) \\ &\geq -M\rho(x,t) \end{aligned}$$

$\rho(0,t) = 0, \rho(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}\rho(x,t)\mid_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho(x,t) \geq 0$ on $\overline{Q_T}$. Therefore $t^{1-q}v^{k+1}(x,t) \leq t^{1-q}u(x,t)$ on $\overline{Q_T}$. Similarly, we can show that $t^{1-q}u(x,t) \leq t^{1-q}w^{k+1}(x,t)$ on $\overline{Q_T}$.

Let $\rho(x,t) = u(x,t) - w^{k+1}(x,t)$. Then from hypothesis, we get

$$\begin{aligned} {}^{c}\partial_t^q \rho(x,t) - k\frac{\partial^2\rho(x,t)}{\partial x^2} &= {}^{c}\partial_t^q u(x,t) - k\frac{\partial^2 u(x,t)}{\partial x^2} - \left[{}^{c}\partial_t^q w^{k+1}(x,t) - k\frac{\partial^2 w^{k+1}(x,t)}{\partial x^2}\right] \\ &\geq f(x,t,u(x,t))) - [f(x,t,w^{k+1}(x,t))] \\ &\geq -M(u(x,t) - w^{k+1}(x,t)) \\ &\geq -M\rho(x,t) \end{aligned}$$

$\rho(0,t) = 0, \rho(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}\rho(x,t) \mid_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho(x,t) \geq 0$ on $\overline{Q_T}$. Therefore $t^{1-q}u(x,t) \leq t^{1-q}w^{k+1}(x,t)$ on $\overline{Q_T}$.

Hence we constructed the monotone sequence $\{v^n(x,t)\}$, $\{w^n(x,t)\}$ of lower and upper solutions of integral representation of linear problem and an appropriate computation process, we show that the sequences $\{t^{1-q}v^n(x,t)\}$ and $\{t^{1-q}w^n(x,t)\}$ are uniformly bounded and equicontinuous. Using the Ascoli-Arzela theorem, we obtain subsequences of $\{t^{1-q}v^n(x,t)\}$ and $\{t^{1-q}w^n(x,t)\}$ which converge uniformly and monotonically on $\overline{Q_T}$. Since the sequences $\{t^{1-q}v^n(x,t)\}$ and $\{t^{1-q}w^n(x,t)\}$ are monotone, the entire sequence $\{t^{1-q}v^n(x,t)\}$ and $\{t^{1-q}w^n(x,t)\}$ converges to $t^{1-q}\rho(x,t)$ and $t^{1-q}\gamma(x,t)$ respectively. From this it follows that

$$\begin{aligned} t^{1-q}v^0(x,t) \leq t^{1-q}v^1(x,t) \leq ... \leq t^{1-q}v^n(x,t) \leq ... \leq t^{1-q}\rho(x,t) \leq t^{1-q}u(x,t) \\ \leq t^{1-q}\gamma(x,t) \leq ... \leq t^{1-q}w^n(x,t) \leq ... \leq t^{1-q}w^0(x,t) \quad on \quad \overline{Q_T}. \end{aligned}$$

Consequently, $\rho(x,t)$ and $\gamma(x,t)$ are coupled minimal and maximal solutions of (3.6) since

$$t^{1-q}v^0(x,t) \leq t^{1-q}\rho(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}\gamma(x,t) \leq t^{1-q}w^0(x,t) \quad on \quad \overline{Q_T}.$$

We prove the uniqueness of the solution of (3.6)in the following.

Theorem 4.2 Let all the assumptions of Theorem 4.1 hold. Further, let $f(x,t,u)$ satisfy the one sided Lipschitz condition of the form

$$f(x,t,u_1) - f(x,t,u_2) \leq M(u_1 - u_2), \quad M > 0.$$

Then the solution $u(x,t)$ of (3.6) exists and is unique.

Proof. We have already proved (ρ, γ) are minimal and maximal solutions of (3.6) on $\overline{Q_T}$. Hence it is enough to show that $\gamma(x,t) \leq \rho(x,t)$ *on* $\overline{Q_T}$.
It is known from Theorem(4.1) that $\gamma(x,t) \leq \rho(x,t)$ *on* $\overline{Q_T}$.
Let $p(x,t) = \gamma(x,t) - \rho(x,t)$. By the hypothesis, we get

$$\begin{aligned} {}^{c}\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2} &= {}^{c}\partial_t^q \gamma(x,t) - k\frac{\partial^2\gamma(x,t)}{\partial x^2} - \left[{}^{c}\partial_t^q \rho(x,t) - k\frac{\partial^2\rho(x,t)}{\partial x^2}\right] \\ &\leq f(x,t,\gamma(x,t)) - [f_i(x,t,\rho(x,t))] \\ &\leq M \mid \gamma(x,t) - \rho(x,t) \mid \\ &\leq M \mid p(x,t) \mid. \end{aligned}$$

$p(0,t)=0, p(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t)\leq 0$. This proves that $\gamma(x,t)=\rho(x,t)=u(x,t)$ on $\overline{Q_T}$ and proof is complete.

V. CONCLUSION

In this work, initially we have obtained the maximal principle and comparison theorem relative to the non-linear weakly coupled Caputo fractional reaction-diffusion equations of (3.6) on QT . Using the comparison result as a tool, we have developed a generalized monotone method for the nonlinear Caputo fractional reaction-diffusion equations of (3.6). Generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (3.6). We have also proved the uniqueness solution of u(x, t) of (3.6).

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