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A Glimpse On Finding Integer Solutions to Quaternary Quadratic Diophantine Equation $xy+4zw=(x+y)(z+w)$

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Abstract: *This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by* $xy+4zw=(x+y)(z+w)$ *.Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduce the quadratic equation with four unknowns to solvable Pythagorean equation. A few relations among the solutions are presented.*

Keywords: Quaternary quadratic equation, Homogeneous quadratic equation, Integer solutions

Notations:

$$t_{3,n}=\frac{n(n+1)}{2}$$

$$P_n^3=\frac{n(n+1)(n+2)}{6}$$

$$P_n^4=\frac{n(n+1)(2n+1)}{6}$$

$$P_n^5=\frac{n^2(n+1)}{2}$$

I. INTRODUCTION

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety .In particular, quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [1-11] for second degree diophantine equations with three and two unknowns representing different geometrical figures. In [12] ,Quadratic equation with four unknowns is considered for obtaining integer solutions.This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by $xy+4zw=(x+y)(z+w)$ .Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduce the quadratic equation with four unknowns to solvable Pythagorean equation. A few relations among the solutions are presented.

II. METHOD OF ANALYSIS

The polynomial equation of second degree with four unknowns to be solved is

$$xy + 4zw = (x+y)(z+w) \quad (1)$$

The procedure to obtain patterns of integer solutions to (1) is as below:

Procedure 1

The option

$$x = u+p, y = u-p, z = u+q, w = u-q, u \neq \pm p, \pm q \quad (2)$$

in (1) gives

$$u^2 = p^2 + (2q)^2 \quad (3)$$

which is in the form of Pythagorean equation satisfied by

$$q = rs, p = r^2 - s^2, u = r^2 + s^2, r > s > 0 \quad (4)$$

From (2) , we get

$$x = 2r^2, y = 2s^2, z = r^2 + s^2 + rs, w = r^2 + s^2 - rs \quad (5)$$

Thus, (5) satisfies (1).

Observations:

$2(x+y) + 4(z-w)$ is a perfect square

$xy = (z-w)^2$

$xy(z-w)$ is a cubical integer

$(x-y)^2 + 4(z-w)^2 = (z+w)^2$

Note 1:

It is seen that , (3) is satisfied by

$$p == 2rs, \; q = \frac{(r^2 - s^2)}{2}, u = r^2 + s^2$$

As our aim is to obtain integer solutions , it is seen that the values of r ,s are of the same parity.

When

$$r = 2R, s = 2S, R > S > 0$$

the corresponding integer solutions to (1) are given by

$$x = 4(R+S)^2, y = 4(R-S)^2, z = 6R^2 + 2S^2, w = 2R^2 + 6S^2$$

Observations :

$x + y + z + w$ is written as sum of two squares

When R ,S are taken as legs of Pythagorean triangle , $x + y + z + w$ is a perfect square

When $R = 2m(m^2+n^2)$,$S = 2n(m^2+n^2)$, $x + y + z + w$ is a cubical integer

When R , S represent hypotenuse and a leg of Pythagorean triangle respectively, the product xy is a quartic integer

When R ,S represent hypotenuse and a leg of Pythagorean triangle respectively, then $z - w$ is a perfect square

$R = S+1 \Rightarrow x - y = 32\, t_{3,S}$

$R = S(S+1) \Rightarrow x - y = 32\, P_S^5$

$R = (S+1)(S+2) \Rightarrow x - y = 96\, P_S^3$

$R = (S+1)(2S+1) \Rightarrow x - y = 96\, P_S^4$

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When

$$r = 2R + 1, s = 2S + 1$$

the corresponding integer solutions to (1) are given by

$$x = 4(R + S + 1)^2, y = 4(R - S)^2$$

$$z = 6(R^2 + R + S^2 + S) + 2$$

$$w = 2(R^2 + R + S^2 + S) + 2$$

Observations:

$x + y + z + w$ is written as sum of two squares

$z + w - x$ is a perfect square

$z - w = 8(t_{3,R} + t_{3,S})$

$z - w + 2$ is written as sum of two squares

$x + y + 5w = 3z + 8$

Note 2

Rewrite (3) as

$$p^2 + (2q)^2 = u^2 * 1 \quad (6)$$

Assume

$$u = (r^2 + s^2)^2 [a^2 + (2b)^2] \quad (7)$$

Express the integer 1 in (6) as

$$1 = \frac{(r^2 - s^2 + i2rs)(r^2 - s^2 - i2rs)}{(r^2 + s^2)^2} \quad (8)$$

Substituting (7) & (8) in (6) and using factorization, we have

$$p + i2q = \frac{(r^2 - s^2 + i2rs)}{(r^2 + s^2)} (r^2 + s^2)^2 (a + i2b)^2$$

from which, we get

$$\begin{aligned} p &= (r^2 + s^2)[(r^2 - s^2)(a^2 - 4b^2) - 8rsab], \\ q &= (r^2 + s^2)[(r^2 - s^2)(2ab) + rs(a^2 - 4b^2)]. \end{aligned} \quad (9)$$

Substituting (7) & (9) in (2) , the corresponding integer solutions to (1) are obtained.

Procedure 2

Introduction of the transformations

$$x = 2u + 2p, y = 2u - 2p, z = p + q, w = p - q \quad (10)$$

in (1) leads to the Pythagorean equation

$$(u - p)^2 = q^2 + p^2$$

Employing the most cited solutions of the above Pythagorean equation in (10),the corresponding integer solutions to (1) are obtained.

Procedure 3

The option

$$x = u + v, y = u - v, z = p + q, w = p - q, u \neq \pm v, p \pm q \quad (11)$$

in (1) gives

$$(u - 2p)^2 = v^2 + (2q)^2 \quad (12)$$

which is in the form of Pythagorean equation satisfied by

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$$q = rs, v = r^2 - s^2, u = 2p + r^2 + s^2, r > s > 0 \quad (13)$$

From (2) , we get

$$x = 2p + 2r^2, y = 2p + 2s^2, z = p + rs, w = p - rs \quad (14)$$

Thus, (14) satisfies (1).

Note 3:

It is seen that , (12) is satisfied by

$$v == 2rs,\ q = \frac{(r^2 - s^2)}{2}, u = 2p + r^2 + s^2$$

As our aim is to obtain integer solutions , it is seen that the values of r ,s are of the same parity.

When

$$r = 2R, s = 2S, R > S > 0$$

the corresponding integer solutions to (1) are given by

$$x = 2p + 4(R+S)^2, y = 2p + 4(R-S)^2, z = p + 2(R^2 - S^2), w = p - 2(R^2 - S^2)$$

When

$$r = 2R + 1, s = 2S + 1$$

the corresponding integer solutions to (1) are given by

$$x = 2p + 4(R+S+1)^2, y = 2p + 4(R-S)^2$$

$$z = p + 2(R^2 + R - S^2 - S)$$

$$w = p - 2(R^2 + R - S^2 - S)$$

III. CONCLUSION

Plenty of solutions in integers are presented in this paper for the quaternary homogeneous quadratic equation given by $xy + 4zw = (x+y)(z+w)$ through employing substitution technique and factorization method. As quadratic equations (homogeneous or non-homogeneous) are plenty, one may search for patterns of integer solutions to other choices of multivariable quadratic equations.

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