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Study on U^h - Birecurrent Finsler Space

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Abstract: *In this paper, we introduce a Finsler space which the curvature tensor* U^i_{jkh} *satisfies the birecurrence property in sense of Cartan. We obtain the necessary and sufficient condition for some tensors to be birecurrent. The relationship between the curvature tensor* U^i_{jkh} *and Douglas tensor* D^i_{jkh} *have been studied. Also, some results in a projection on indicatrix with respect to Cartan connection have been discussed.*

Keywords: U^h-birecurrent space, Projection on indicatrix

I. Introduction

The definition for normal projective tensor N^i_{jkh} and connection coefficients Π^i_{jk} for it introduced by Yano [10].The definition for Douglas tensor and some types of it studied by Bácso and Matsumoto [14].Dikshit [15] defind a Finsler space which the normal projective curvature tensor is birecurrent in sense of Berwald. Misra and Meher [13] considered a space equipped with normal projective connection coefficients Π^i_{jk} whose curvature tensor N^i_{jkh} is recurrent with respect to normal projective connection coefficients Π^i_{jk} and called it an $RNP-$Finsler space. Ali [11] and Hanballa [5] studied the birecurrence property for Cartan's fourth curvature tensor in sense of Cartan and Berwald, respectively.Otman [6] introduced $\mathcal{BP}-$birecurrent space.Alaa et al. [2] studied some tensors whichbebirecurrent. Saleem [3]discussed the tensor U^i_{jkh} which satisfies the birecurrence property in sense of Berwald.Additionally, Alaa et al. [1] andHanballa [5] studied the projection on indicatrix for some tensors with respect to Berwald connection and Cartan's connection, respectively.

Let F_n be an $n-$dimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [9].The unit vector l^i and the associative vector l_i with the direction of y^i are given by

(1.1) a) $l^i = y^i/F$ and b) $l_i = y_i/F$,

where F is a fundamental metric function which be positive homogeneous of degree one in y^i. The vectors y_i and y^i satisfy

(1.2) $y_i y^i = F^2$.

Cartan $h-$covariant differentiation (Cartan's second kind covariant differentiation) with respect to x^l is given by [8, 9]

$$X^i_{|l} = \partial_l X^i - (\dot{\partial}_r X^i) G^r_l + X^r \Gamma^{*i}_{rl} .$$

The $h-$covariant derivative of the vector y^i and the metric function F are vanish identically*i.e.*

(1.3) a) $y^i_{|k} = 0$ and b) $F_{|l} = 0$.

For an arbitrary tensor field T_h^i, the $h-$covariant differentiation with respect to x^l which defined above, commute with the partial differentiation with respect to y^j according to

(1.4) $\dot{\partial}_j\left(T^i_{h|l}\right)-\left(\dot{\partial}_j T^i_h\right)_{|l}=T^r_h\left(\dot{\partial}_j\Gamma^{*i}_{lr}\right)-T^i_r\left(\dot{\partial}_j\Gamma^{*r}_{lh}\right)-\left(\dot{\partial}_r T^i_h\right)P^r_{jl}$,

where

(1.5) $P^r_{jl}=\left(\dot{\partial}_j\Gamma^{*r}_{hl}\right)y^h=\Gamma^{*r}_{jhl}y^h$,

and

(1.6) $P^r_{jl}y^l=0$.

II. Preliminaries

In this section, we introduce some conditions and definitions which are needed in this paper. The normal projective tensor N^i_{jkh} is defined as follows [10]

$$N^i_{jkh}=\dot{\partial}_j\Pi^i_{kh}+\Pi^i_{rjh}\Pi^r_{ks}y^s+\Pi^i_{rh}\Pi^r_{kj}-k\,|\,h,$$

where $\Pi^i_{jkh}=G^i_{jkh}-\frac{1}{n+1}\left(\delta^i_j G^r_{jkr}+y^i G^r_{jkhr}\right)$ and $\Pi^i_{jkh}=\dot{\partial}_j\Pi^i_{kh}$,

Π^i_{jkh} consider the components of a tensor.

The normal projective connection coefficients Π^i_{jk} is positively homogeneous of degree zero in y^i and symmetric in their lower indices is defined by [10]

$\Pi^i_{jk}=G^i_{jk}-y^i G^r_{jkr}$,

where $G^i_{jk}=\dot{\partial}_j G^i_k$.

Yano denoted for the tensor Π^i_{jkh} by U^i_{jkh} which is defined by [10]

(2.1) $U^i_{jkh}=G^i_{jkh}-\frac{1}{n+1}\left(\delta^i_j G^r_{jkr}+y^i G^r_{jkhr}\right)$

and

(2.2) $G^r_{jkhr}=\dot{\partial}_j G^r_{khr}$,

where G^i_{jkh} consider a connection of the curvature tensor U^i_{jkh}. This tensor is homogeneous of degree -1 in y^i and symmetric in its last two indices, i.e.

$U^i_{jkh}=U^i_{jhk}$.

Also, this tensor satisfies the following

(2.3) a) $U^r_{jrh}=U^r_{jkr}=G^r_{jkr}$, b) $U^i_{jkh}y^j=0$ and c) $U^i_{jkh}y^h=U^i_{jhk}y^h=U^i_{jk}$,

where the torsion tensor U^i_{jk} satisfies

(2.4) a) $U^i_{jk}=U^i_{kj}$, b) $U^r_{jr}=G^r_{kr}$ and c) $U^i_{jk}y^k=U^i_{kj}y^k=G^i_j$,

where the tensor G^i_j is deviation tensor which be homogeneous of degree 1 in y^i and satisfy

(2.5) $G^i_j y^j=2G^i$,

where G^i is positively homogeneous of degree 2 in y^i.

The $U-$Ricci tensor U_{jk} satisfies the following [9]

(2.6) a) $U^r_{rkh} = U_{kh}$ and b) $U_{jk} = \frac{2}{n+1} G_{jk}$,

where the tensor G_{jk} is components of the projective connection coefficients.

The Douglas tensor is given by [7, 14]

(2.7) $D^i_{jkh} = U^i_{jkh} - \frac{1}{2}\left(\delta^i_j U_{kh} + \delta^i_k U_{jh}\right)$.

Which satisfies the following

(2.8) $D^i_{jkh} y^j = D^i_{kjh} y^j = D^i_{khj} y^j = 0$.

Definition 2.1. Let the current coordinates in the tangent space at the point x_0 be x^i, then the indicatrix I_{n-1} is a hypersurface defined by [1, 4,12]

$F\left(x_0\ ,\ x^i\right) = 1$ or by the parametric form defined by $x^i = x^i\left(u^a\right)$, $a = 1, 2, \ldots, n-1$.

Definition 2.2. The projection of any tensor T^i_j on indicatrix I_{n-1} is given by [1, 9]

(2.9) $p\ .T^i_j = T^a_b\, h^i_a\, h^b_j$,

where

(2.10) $h^i_c = \delta^i_c - l^i l_c$.

The projection of the vector y^i, the unit vector l^i and the metric tensor g_{ij} on the indicatrix are given by

$p.y^i = 0$, $p.l^i = 0$ and $p.g_{ij} = h_{ij}$,

where $h_{ij} = g_{ij} - l_i l_j$.

Lately, Saleem and Abdallah [4] introduced the U^h-recurrent space, i.e. the tensor U^i_{jkh} is characterized by the condition

(2.11) $U^i_{jkh|l} = \lambda_l U^i_{jkh}, U^i_{jkh} \neq 0$,

where λ_l is non-zero covariant vector field.

III. U^h-Birecurrent Space

In this section, we introduce a Finsler space which U^i_{jkh} bebirecurrent in sense of Cartan. Also, we find the condition for some tensors which satisfy the birecurrence property.

Definition 3.1. A Finsler space F_n which the tensor U^i_{jkh} satisfies the birecurrence property, i.e. characterized by condition

(3.1) $U^i_{jkh|l|m} = a_{lm} U^i_{jkh}, U^i_{jkh} \neq 0$,

where a_{lm} is non-zero covariant tensor field. This space will be called a U^h-*birecurrent space* and denoted it briefly by $U^h - BRF_n$.

Let us consider a $U^h - BRF_n$. Differentiating (2.11) covariantly with respect to x^m in the sense of Cartan and using (1.3b), we get

$U^i_{jkh|l|m} = \lambda_{l|m} U^i_{jkh} + \lambda_l U^i_{jkh|m}$.

In view of (2.11), above equation becomes

$U^i_{jkh|l|m} = \lambda_{l|m} U^i_{jkh} + \lambda_l \lambda_m U^i_{jkh}$

which can be written as

$U^i_{jkh|l|m} = a_{lm} U^i_{jkh}$, $U^i_{jkh} \neq 0$,

where

$a_{lm} = \lambda_{l|m} + \lambda_l \lambda_m$.

Thus, we conclude

Theorem 3.1. *Every $U^h - RF_n$ is $U^h - BRF_n$ for the recurrence vector field satisfies $\lambda_{l|m} + \lambda_l \lambda_m \neq 0$.*

Transvecting (3.1) by y^h, using (2.3c) and (1.3a), we get

(3.2) $U^i_{jk|l|m} = a_{lm} U^i_{jk}$.

Contracting the indices i and j in (3.1) and using (2.6a), we get

(3.3) $U_{kh|l|m} = a_{lm} U_{kh}$.

Contracting of the indices i and h in (3.1) and using (2.3a), we get

(3.4) $G^r_{jkr|l|m} = a_{lm} G^r_{jkr}$.

Contracting the indices i and k in (3.2) and using (2.4b), we get

(3.5) $G^r_{jr|l|m} = a_{lm} G^r_{jr}$.

Transvecting (3.2) by y^k, using (2.4c) and (1.3a), we get

(3.6) $G^i_{j|l|m} = a_{lm} G^i_j$.

Transvecting (3.6) by y^j, using (2.5) and (1.3a), we get

(3.7) $G^i_{|l|m} = a_{lm} G^i$.

Thus, we conclude

Theorem 3.2. *The $hv -$ torsion tensor U^i_{jk}, $hv -$ Ricci tensor U_{jk}, tensor G^r_{jkr}, torsion tensor G^r_{jr}, the deviation G^i_j and the vector G^i of $U^h - BRF_n$ are Birecurrent.*

Differentiating (2.7) twice covariantly with respect to x^l and x^m in sense of Cartan, we get

(3.8) $D^i_{jkh|l|m} = U^i_{jkh|l|m} - \frac{1}{2}\left(\delta^i_j U_{kh|l|m} + \delta^i_k U_{jh|l|m}\right)$.

Using (3.1) and (3.3) in (3.8), we get

$$D^i_{jkh|l|m} = a_{lm}\left\{U^i_{jkh} - \frac{1}{2}\left(\delta^i_j U_{kh} + \delta^i_k U_{jh}\right)\right\}.$$

Using (2.7) in above equation, we get

(3.9) $D^i_{jkh|l|m} = a_{lm} D^i_{jkh}$.

Thus, we conclude

Theorem 3.3. *in $U^h - BRF_n$, the Douglas tensor D^i_{jkh} is birecurrent.*

If the Douglas tensor D^i_{jkh} and $U-$Ricci tensor U_{kh} are birecurrent in Finsler space, then this space is necessaryto be U^h-BRF_n. This will be seen as follows:

Equation (3.8) can be written as

(3.10) $$U^i_{jkh|l|m} = D^i_{jkh|l|m} + \frac{1}{2}\left(\delta^i_j U_{kh|l} + \delta^i_k U_{jh|l}\right).$$

From (3.9) and (3.3), we have the Douglas tensor D^i_{jkh} and $U-$Ricci tensor U_{kh} behave as birecurrent, then above equation become as

$$U^i_{jkh|l|m} = a_{lm}\left\{D^i_{jkh} + \frac{1}{2}\left(\delta^i_j U_{kh} + \delta^i_k U_{jh}\right)\right\}.$$

Using (2.8) in above equation, we get

$$U^i_{jkh|l|m} = a_{lm} U^i_{jkh}.$$

Thus, we conclude

Theorem 3.4. *In Finslerspace F_n, if the Douglas tensor and $U-$Ricci tensor are birecurrent, then this space is necessarily considered U^h-BRF_n.*

IV. Necessary and Sufficient Condition for some Tensors to be Birecurrent Space

In thus section, we find thenecessary and sufficient condition for some tensors to be birecurrentin U^h-BRF_n.Let us consider a U^h-BRF_n. Differentiating (3.4) partially with respect to y^h, we get

(4.1) $$\dot{\partial}_h G^r_{jkr|l|m} = \left(\dot{\partial}_h a_{lm}\right) G^r_{jkr} + a_{lm}\left(\dot{\partial}_h G^r_{jkr}\right).$$

Using commutation formula exhibited by (1.4) for G^r_{jkr} and (2.2) in (4.1), we get

(4.2) $$\left(\dot{\partial}_h G^r_{jkr|l}\right)_{|m} - G^r_{skr|l}\left(\dot{\partial}_h \Gamma^{*s}_{jm}\right) - G^r_{jsr|l}\left(\dot{\partial}_h \Gamma^{*s}_{km}\right) - (\dot{\partial}_s G^r_{jkr|l}) P^s_{hm} = \left(\dot{\partial}_h a_{lm}\right) G^r_{jkr} + a_{lm} G^r_{jkhr}$$

Again applying commutation formula exhibited by (1.4) for G^r_{jkr} and using (2.2) in (4.2), we get

$$\left\{G^r_{jkhr|l} - G^r_{skr}\left(\dot{\partial}_h \Gamma^{*s}_{jl}\right) - G^r_{jsr}\left(\dot{\partial}_h \Gamma^{*s}_{kl}\right) - G^r_{jksr} P^s_{hl}\right\}_{|m} - G^r_{skr|l}\left(\dot{\partial}_h \Gamma^{*s}_{jm}\right) - G^r_{jsr|l}\left(\dot{\partial}_h \Gamma^{*s}_{km}\right)$$
$$-\left\{G^r_{jkrs|l} - G^r_{tkr}\left(\dot{\partial}_s \Gamma^{*t}_{jl}\right) - G^r_{jsr}\left(\dot{\partial}_s \Gamma^{*t}_{kl}\right) - G^r_{jktr} P^t_{hl}\right\} P^s_{hm} = \left(\dot{\partial}_h a_{lm}\right) G^r_{jkr} + a_{lm} G^r_{jkhr},$$

which be rewritten as

(4.3) $$G^r_{jkhr|l|m} - \left\{G^r_{skr}\left(\dot{\partial}_h \Gamma^{*s}_{jl}\right) + G^r_{jsr}\left(\dot{\partial}_h \Gamma^{*s}_{kl}\right) + G^r_{jksr} P^s_{hl}\right\}_{|m} - G^r_{skr|l}\left(\dot{\partial}_h \Gamma^{*s}_{jm}\right) - G^r_{jsr|l}\left(\dot{\partial}_h \Gamma^{*s}_{km}\right)$$
$$-\left\{G^r_{jkrs|l} - G^r_{tkr}\left(\dot{\partial}_s \Gamma^{*t}_{jl}\right) - G^r_{jsr}\left(\dot{\partial}_s \Gamma^{*t}_{kl}\right) - G^r_{jktr} P^t_{hl}\right\} P^s_{hm} = \left(\dot{\partial}_h a_{lm}\right) G^r_{jkr} + a_{lm} G^r_{jkhr}.$$

This shows that

(4.4) $$G^r_{jkhr|l|m} = a_{lm} G^r_{jkhr}$$

if and only if

(4.5) $$\left\{G^r_{skr}\left(\dot{\partial}_h \Gamma^{*s}_{jl}\right) + G^r_{jsr}\left(\dot{\partial}_h \Gamma^{*s}_{kl}\right) + G^r_{jksr} P^s_{hl}\right\}_{|m} + G^r_{skr|l}\left(\dot{\partial}_h \Gamma^{*s}_{jm}\right) + G^r_{jsr|l}\left(\dot{\partial}_h \Gamma^{*s}_{km}\right)$$
$$+\left\{G^r_{jkrs|l} - G^r_{tkr}\left(\dot{\partial}_s \Gamma^{*t}_{jl}\right) - G^r_{jsr}\left(\dot{\partial}_s \Gamma^{*t}_{kl}\right) - G^r_{jktr} P^t_{hl}\right\} P^s_{hm} - \left(\dot{\partial}_h a_{lm}\right) G^r_{jkr} = 0.$$

Thus, we conclude

Theorem 4.1. *The tensor* G^r_{jkhr} *in* U^h-BRF_n *behave as birecurrent if and only if (4.5) holds.*

Transvecting (4.3) by y^l , using (1.5), (1.6) and (1.3a), we get

$$y^l\ G^r_{jkhr|l|m}=a_{lm}y^l\ G^r_{jkhr}$$

if and only if

$$\text{(4.6)}\qquad \left\{G^r_{skr}\Gamma^{*s}_{hj}+G^r_{jsr}\Gamma^{*s}_{hk}\right\}_{|m}+y^lG^r_{skr|l}\left(\dot{\partial}_h\Gamma^{*s}_{jm}\right)+y^lG^r_{jsr|l}\left(\dot{\partial}_h\Gamma^{*s}_{km}\right)$$

$$+\left\{y^lG^r_{jkrs|l}-G^r_{tkr}\Gamma^{*t}_{sj}-G^r_{jsr}\Gamma^{*t}_{sk}\right\}P^s_{hm}-y^l\left(\dot{\partial}_ha_{lm}\right)G^r_{jkr}=0\,.$$

Thus, we conclude

Theorem 4.2. *In* U^h-BRF_n *, the directional derivative of the tensor* G^r_{jkhr} *in the directional of* y^m *is proportional to the tensor* G^r_{jkhr} *if and only if (4.6) holds.*

Again, transvecting (4.3) by y^m , using (1.5), (1.6) and (1.3a), we get

$$y^m\ G^r_{jkhr|l|m}=a_{lm}y^m\ G^r_{jkhr}$$

if and only if

$$\text{(4.7)}\qquad \left\{G^r_{skr}\left(\dot{\partial}_h\Gamma^{*s}_{jl}\right)+G^r_{jsr}\left(\dot{\partial}_h\Gamma^{*s}_{kl}\right)+G^r_{jksr}P^s_{hl}\right\}_{|m}y^m$$

$$+G^r_{skr|l}\Gamma^{*s}_{hj}+G^r_{jsr|l}\Gamma^{*s}_{hk}-y^m\left(\dot{\partial}_ha_{lm}\right)G^r_{jkr}=0\,.$$

Thus, we conclude

Theorem 4.3. *In* U^h-BRF_n *, the directional derivative of the tensor* G^r_{jkhr} *in the directional of* y^l *is proportional to the tensor* G^r_{jkhr} *if and only if (4.7) holds.*

Differentiating (2.1) twice covariantly with respect to x^l and x^m in sense of Cartan, we get

$$U^i_{jkh|l|m}=G^i_{jkh|l|m}-\frac{1}{n+1}\left(\delta^i_jG^r_{jkr|l|m}+y^iG^r_{jkhr|l|m}\right).$$

Using (3.1) in above equation, we get

$$a_{lm}U^i_{jkh}=G^i_{jkh|l|m}-\frac{1}{n+1}\left(\delta^i_jG^r_{jkr|l|m}+y^iG^r_{jkhr|l|m}\right).$$

Using (2.1), and (4.3) in above equation, we get

$$\text{(4.8)}\qquad G^i_{jkh|l|m}-a_{lm}G^i_{jkh}=\frac{1}{n+1}y^i[\left\{G^r_{skr}\left(\dot{\partial}_h\Gamma^{*s}_{jl}\right)+G^r_{jsr}\left(\dot{\partial}_h\Gamma^{*s}_{kl}\right)+G^r_{jksr}P^s_{hl}\right\}_{|m}+G^r_{skr|l}\left(\dot{\partial}_h\Gamma^{*s}_{jm}\right)$$

$$+G^r_{jsr|l}\left(\dot{\partial}_h\Gamma^{*s}_{km}\right)+\left\{G^r_{jkrs|l}-G^r_{tkr}\left(\dot{\partial}_s\Gamma^{*t}_{jl}\right)-G^r_{jsr}\left(\dot{\partial}_s\Gamma^{*t}_{kl}\right)-G^r_{jktr}P^t_{hl}\right\}P^s_{hm}-\left(\dot{\partial}_ha_{lm}\right)G^r_{jkr}].$$

Therefore

$$G^i_{jkh|l|m}=a_{lm}G^i_{jkh}$$

if and only if (4.5) is holds.

Thus, we conclude

Theorem 4.4. *The tensor* G^i_{jkh} *of* U^h-BRF_n *is birecurrent if and only if (4.5) holds.*

Transvecting (4.8) by y_i and using (1.2), we get

$$\frac{y_i}{F^2}\left(G^i_{jkh|l|m}-a_{lm}G^i_{jkh}\right)=\frac{1}{n+1}[\left\{G^r_{skr}\left(\dot{\partial}_h\Gamma^{*s}_{jl}\right)+G^r_{jsr}\left(\dot{\partial}_h\Gamma^{*s}_{kl}\right)+G^r_{jksr}P^s_{hl}\right\}_{|m}+G^r_{skr|l}\left(\dot{\partial}_h\Gamma^{*s}_{jm}\right)$$

$+G^{r}_{jsr|l}\left(\dot{\partial}_{h}\Gamma^{*s}_{km}\right)+\left\{G^{r}_{jkrs|l}-G^{r}_{tkr}\left(\dot{\partial}_{s}\Gamma^{*t}_{jl}\right)-G^{r}_{jsr}\left(\dot{\partial}_{s}\Gamma^{*t}_{kl}\right)-G^{r}_{jktr}P^{t}_{hl}\right\}P^{s}_{hm}-\left(\dot{\partial}_{h}a_{lm}\right)G^{r}_{jkr}].$

Using (4.8) and (1.1) in above equation, we get

(4.9) $\quad G^{i}_{jkh|l|m}-a_{lm}G^{i}_{jkh}=l^{i}l_{r}\left(G^{r}_{jkh|l|m}-a_{lm}G^{r}_{jkh}\right).$

In view of (1.1), above equation can be written as

$\left(G^{i}_{jkh}-l^{i}l_{r}G^{r}_{jkh}\right)_{|l|m}=a_{lm}\left(G^{r}_{jkh|l|m}-l^{i}l_{r}G^{r}_{jkh}\right).$

Thus, we conclude

Theorem 4.5. *The tensor $\left(G^{r}_{jkh|l|m}-l^{i}l_{r}G^{r}_{jkh}\right)$ of $U^{h}-BRF_{n}$ is birecurrent.*

From (4.9), we conclude

Transvecting (4.8) by y^{l}, using (1.5), (1.6) and (1.3a), we get

$$y^{l}\left(G^{i}_{jkh|l|m}-a_{lm}G^{i}_{jkh}\right)=\frac{y^{i}}{n+1}[\left\{G^{r}_{skr}\Gamma^{*s}_{hj}+G^{r}_{jsr}\Gamma^{*s}_{hk}\right\}_{|m}+y^{l}G^{r}_{skr|l}\left(\dot{\partial}_{h}\Gamma^{*s}_{jm}\right)+y^{l}G^{r}_{jsr|l}\left(\dot{\partial}_{h}\Gamma^{*s}_{km}\right)$$

$$+\left\{G^{r}_{jkrs|l}-G^{r}_{tkr}\left(\dot{\partial}_{s}\Gamma^{*t}_{jl}\right)-G^{r}_{jsr}\left(\dot{\partial}_{s}\Gamma^{*t}_{kl}\right)\right\}P^{s}_{hm}y^{l}-y^{l}\left(\dot{\partial}_{h}a_{lm}\right)G^{r}_{jkr}],$$

which implies

$y^{l}G^{i}_{jkh|l|m}=a_{lm}y^{l}G^{i}_{jkh}$

if and only if

(4.10) $\quad\left\{G^{r}_{skr}\Gamma^{*s}_{hj}+G^{r}_{jsr}\Gamma^{*s}_{hk}\right\}_{|m}+y^{l}G^{r}_{skr|l}\left(\dot{\partial}_{h}\Gamma^{*s}_{jm}\right)+y^{l}G^{r}_{jsr|l}\left(\dot{\partial}_{h}\Gamma^{*s}_{km}\right)$

$+\left\{G^{r}_{jkrs|l}-G^{r}_{tkr}\left(\dot{\partial}_{s}\Gamma^{*t}_{jl}\right)-G^{r}_{jsr}\left(\dot{\partial}_{s}\Gamma^{*t}_{kl}\right)\right\}P^{s}_{hm}y^{l}-y^{l}\left(\dot{\partial}_{h}a_{lm}\right)G^{r}_{jkr}=0.$

Thus, we conclude

Theorem 4.6. *In $U^{h}-BRF_{n}$, the directional derivative of the tensor G^{i}_{jkh} in the directional of y^{m} is proportional to the tensor G^{i}_{jkh} if and only if (4.10) holds.*

Again, transvecting (4.8) by y^{m}, using (1.5), (1.6) and (1.3a), we get

$y^{m}\;G^{i}_{jkh|l|m}=a_{lm}y^{m}\;G^{i}_{jkh}$

if and only if

(4.11) $\left\{G^{r}_{skr}\left(\dot{\partial}_{h}\Gamma^{*s}_{jl}\right)+G^{r}_{jsr}\left(\dot{\partial}_{h}\Gamma^{*s}_{kl}\right)+G^{r}_{jksr}P^{s}_{hl}\right\}_{|m}y^{m}+G^{r}_{skr|l}\Gamma^{*s}_{hj}+G^{r}_{jsr|l}\Gamma^{*s}_{hk}-y^{m}\left(\dot{\partial}_{h}a_{lm}\right)G^{r}_{jkr}=0.$

Thus, we conclude

Theorem 4.7. *In $U^{h}-BRF_{n}$, the directional derivative of the tensor G^{i}_{jkh} in the directional of y^{l} is proportional to the tensor G^{i}_{jkh} if and only if (4.11) holds.*

V. Projection on Indicatrix with Respect to Cartan's Connection

In this section, we prove that, if the tensors behave as birecurrent, then the projection of them are birecurrentin $U^{h}-BRF_{n}$. Also we find the condition for the projection of some tensors on Indicatrix which be birecurrent.

Let us consider a $U^{h}-BRF_{n}$.

We know that, the curvature tensor U^i_{jkh} behaves as birecurrent *i.e.* satisfies (3.1). Now, in view of (2.9), the projection of the curvature tensor U^i_{jkh} on indicatrix is given by

(5.1) $p.U^i_{jkh} = U^a_{bcd} h^i_a h^b_j h^c_k h^d_h$.

Taking covariant derivative of (5.1) with respect to x^l and x^m in sense of Cartan, using (3.1) and the fact that $h^i_{j|l} = 0$, we get

$\left(p.U^i_{jkh}\right)_{|l|m} = a_{lm} U^a_{bcd} h^i_a h^b_j h^c_k h^d_h$.

Using (5.1) in above equation, we get

(5.2) $\left(p.U^i_{jkh}\right)_{|l|m} = a_{lm}\left(p.U^i_{jkh}\right)$.

This shows that $p.U^i_{jkh}$ is birecurrent.

Thus, we conclude

Theorem 5.1. *If the curvature tensor* U^i_{jkh} *behaves as birecurrent, then the projection of itin* $U^h - BRF_n$ *on indicatrix is birecurrent in sense of Cartan.*

We know that, the torsion tensor U^i_{jk} behaves as birecurrent*i.e.* satisfies (3.2). In view of (2.9), the projection of the torsion tensor U^i_{jk} on indicatrix is given by

(5.3) $p.U^i_{jk} = U^a_{bc} h^i_a h^b_j h^c_k$.

Taking covariant derivative of (5.3) with respect to x^l and x^m in sense of Cartan, using (3.2) and the fact that $h^i_{j|l} = 0$, we get

$\left(p.U^i_{jk}\right)_{|l|m} = a_{lm} U^a_{bc} h^i_a h^b_j h^c_k$.

Using (5.3) in above equation, we get

(5.4) $\left(p.U^i_{jk}\right)_{|l|m} = a_{lm}\left(p.U^i_{jk}\right)$.

This shows that $p.U^i_{jk}$ is birecurrent.

Thus, we conclude

Theorem 5.2. *If the torsion tensor* U^i_{jk} *behaves as birecurrent, then the projection of itin* $U^h - BRF_n$ *on indicatrix is birecurrent in sense of Cartan.*

We know that, the $U-$Ricci tensor U_{kh} behaves as birecurrent*i.e.* satisfies (3.3). In view of (2.9), the projection of the $U-$Ricci tensor U_{kh} on indicatrix is given by

(5.5) $p.U_{kh} = U_{ab} h^a_k h^b_h$.

Taking covariant derivative of (5.5) with respect to x^l and x^m in sense of Cartan, using (3.3) and the fact that $h^i_{j|l} = 0$, we get

$\left(p.U_{kh}\right)_{|l|m} = a_{lm} U_{ab} h^a_k h^b_h$.

Using (5.5) in above equation, we get

(5.6) $\left(p.U_{kh}\right)_{|l|m} = a_{lm} U_{ab} h^a_k h^b_h$.

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This shows that $p.U_{jk}$ is birecurrent.

Thus, we conclude

Theorem 5.3. *If the $U-$Ricci tensor U_{kh} behaves as birecurrent, then the projection of itin U^h-BRF_n on indicatrix is birecurrent in sense of Cartan.*

We know that, the Dougles tensor D^i_{jkh} behaves as birecurrent *i.e.* satisfies (3.9). In view of (2.9), the projection of the Dougles tensor D^i_{jkh} on indicatrix is given by

(5.7) $p.D^i_{jkh} = D^a_{bcd} h^i_a h^b_j h^c_k h^d_h$.

Taking covariant derivative of (5.7) with respect to x^l and x^m in sense of Cartan, using (3.9) and the fact that $h^i_{j|l}=0$, we get

$$\left(p.D^i_{jkh}\right)_{|l|m} = a_{lm} D^a_{bcd} h^i_a h^b_j h^c_k h^d_h .$$

Using (5.7) in above equation, we get

(5.8) $\left(p.D^i_{jkh}\right)_{|l|m} = a_{lm}\left(p.D^i_{jkh}\right)$.

This shows that $p.D^i_{jkh}$ is birecurrent.

Thus, we conclude

Theorem 5.4. *If the Douglas tensor D^i_{jkh} behaves as birecurrent, then the projection of itin U^h-BRF_n on indicatrix is birecurrent in sense of Cartan.*

We know that the projection of the curvature tensor U^i_{jkh} on indicatrix behaves as birecurrent *i.e.* satisfied (5.2).

Using (2.9) in (5.2), we get

$$\left(U^a_{bcd} h^i_a h^b_j h^c_k h^d_h\right)_{|l|m} = a_{lm} U^a_{bcd} h^i_a h^b_j h^c_k h^d_h .$$

Using (2.10) in above equation, we get

$$\begin{aligned}
&(U^i_{jkh} - U^i_{jkd} l^d l_h - U^i_{jch} l^c l_k + U^i_{jcd} l^c l_k l^d l_h - U^i_{bkh} l^b l_j + U^i_{bkd} l^b l_j l^d l_h + U^i_{bch} l^b l_j l^c l_k \\
&- U^i_{bcd} l^b l_j l^c l_k l^d l_h - U^a_{jkh} l^i l_a + U^a_{jkd} l^i l_a l^d l_h + U^a_{jch} l^i l_a l^c l_k - U^a_{jcd} l^i l_a l^c l_k l^d l_h \\
&+ U^a_{bkh} l^i l_a l^b l_j - U^a_{bkd} l^i l_a l^b l_j l^d l_h - U^a_{bch} l^i l_a l^b l_j l^c l_k + U^a_{bcd} l^i l_a l^b l_j l^c l_k l^d l_h)_{|l|m} \\
&= a_{lm}(U^i_{jkh} - U^a_{jkd} l^d l_h - U^i_{jch} l^c l_k + U^i_{jcd} l^c l_k l^d l_h - U^i_{bkh} l^b l_j + U^i_{bkd} l^b l_j l^d l_h + U^i_{bch} l^b l_j l^c l_k \\
&- U^i_{bcd} l^b l_j l^c l_k l^d l_h - U^a_{jkh} l^i l_a + U^a_{jkd} l^i l_a l^d l_h + U^a_{jch} l^i l_a l^c l_k - U^a_{jcd} l^i l_a l^c l_k l^d l_h \\
&+ U^a_{bkh} l^i l_a l^b l_j - U^a_{bkd} l^i l_a l^b l_j l^d l_h - U^a_{bch} l^i l_a l^b l_j l^c l_k + U^a_{bcd} l^i l_a l^b l_j l^c l_k l^d l_h).
\end{aligned}$$

Using (1.1a) in above equation, then using (2.3b) and (2.3c) in the resulting equation, we get

$$\begin{aligned}
&(U^i_{jkh} - \frac{1}{F} U^i_{jk} l_h - \frac{1}{F} U^i_{jh} l_k + \frac{1}{F^2} G^i_j l_k l_h - U^a_{jkh} l^i l_a + \frac{1}{F} U^a_{jk} l^i l_a l_h + \frac{1}{F} U^a_{jh} l^i l_a l_k - \frac{1}{F^2} G^a_j l^i l_a l_k l_h)_{|l|m} \\
&= a_{lm}(U^i_{jkh} - \frac{1}{F} U^i_{jk} l_h - \frac{1}{F} U^i_{jh} l_k + \frac{1}{F^2} G^i_j l_k l_h - U^a_{jkh} l^i l_a + \frac{1}{F} U^a_{jk} l^i l_a l_h + \frac{1}{F} U^a_{jh} l^i l_a l_k - \frac{1}{F^2} G^a_j l^i l_a l_k l_h) .
\end{aligned}$$

Now, since the tensors U^i_{jk} and G^i_j are birecurrent, i.e. satisfy (3.2) and (3.6), respectively. Then by using (1.1) and (1.3) in above equation, we have

(5.9) $\left(U^i_{jkh} - U^a_{jkh} l^i l_a\right)_{|l|m} = a_{lm}(U^i_{jkh} - U^a_{jkh} l^i l_a)$.

Thus, we conclude

Theorem 5.5. *If the projection of the tensor* $\left(U^i_{jkh} - U^a_{jkh} l^i l_a\right)$ *on indicatrix is birecurrent, then the space is* $U^h - BRF_n$ *, provided* U^i_{jk} and G^i_j *are birecurrent in sense of Cartan.*

From (5.9), we get

Corollary 5.1. *In* $U^h - BRF_n$ *, the projection of the curvature tensor* U^i_{jkh} *on indicatrix is birecurrent, if and only if* $U^a_{jkh} l_a$ *is birecurrent.*

We know that the projection of the torsion tensor U^i_{jk} on indicatrixbehaves as birecurrent*i.e.* satisfied (5.4).

Using (2.9) in (5.4), we get

$$\left(U^a_{bc} h^i_a h^b_j h^c_k\right)_{|l|m} = a_{lm} U^a_{bc} h^i_a h^b_j h^c_k .$$

Using (2.10) in above equation, we get

$$(U^i_{jk} - U^i_{jc} l^c l_k - U^i_{bk} l^b l_j + U^i_{bc} l^b l_j l^c l_k - U^a_{jk} l^i l_a + U^a_{jc} l^i l_a l^c l_k + U^a_{bk} l^i l_a l^b l_j - U^a_{bc} l^i l_a l^b l_j l^c l_k)_{|l|m}$$
$$= a_{lm}(U^i_{jk} - U^i_{jc} l^c l_k - U^i_{bk} l^b l_j + U^i_{bc} l^b l_j l^c l_k - U^a_{jk} l^i l_a + U^a_{jc} l^i l_a l^c l_k + U^a_{bk} l^i l_a l^b l_j - U^a_{bc} l^i l_a l^b l_j l^c l_k) .$$

Using (1.1a) in above equation, then using (2.4c) and (2.5) in the resulting equation, we get

$$(U^i_{jk} - \frac{1}{F} G^i_j l_k - \frac{1}{F} G^i_k l_j + \frac{1}{F^2} G^i l_j l^c l_k - U^a_{jk} l^i l_a + \frac{1}{F} G^a_j l^i l_a l_k + \frac{1}{F} G^a_k l^i l_a l_j - \frac{1}{F^2} G^a l^i l_a l_j l_k)_{|lm}$$
$$= a_{lm}(U^i_{jk} - \frac{1}{F} G^i_j l_k - \frac{1}{F} G^i_k l_j + \frac{1}{F^2} G^i l_j l^c l_k - U^a_{jk} l^i l_a + G^a_j l^i l_a l_k + G^a_k l^i l_a l_j - \frac{1}{F^2} G^a l^i l_a l_j l_k) .$$

Now, since the tensors G^i_j and G^i are birecurrent, i.e. satisfy (3.6) and (3.7), respectively. Then by using (1.1) and (1.3) in above equation, we have

(5.10) $\left(U^i_{jk} - U^a_{jk} l^i l_a\right)_{|l|m} = a_{lm}\left(U^i_{jk} - U^a_{jk} l^i l_a\right)$.

Thus, we conclude

Theorem 5.6. *If the projection of the tensor* $\left(U^i_{jk} - U^a_{jk} l^i l_a\right)$ *on indicatrix is birecurrent, then the space is* $U^h - BRF_n$ *, provided* G^i_j and G^i *are birecurrent in sense of Cartan.*

From (5.10), we get

Corollary 5.2. *In* $U^h - BRF_n$ *, the projection of the torsion tensor* U^i_{jk} *on indicatrix is birecurrent, if and only if* $U^a_{jk} l_a$ *is birecurrent.*

We know that the projection of the $U-$Ricci tensor U_{kh} on indicatrixbehaves as birecurrent*i.e.* satisfied (5.6).

Using (2.9) in (5.6), we get

$$\left(U_{ab} h^a_k h^b_h\right)_{|l|m} = a_{lm} U_{ab} h^a_k h^b_h .$$

Using (2.10) in above equation, we get

$$\left(U_{kh} - U_{kb} l^b l_h - U_{ah} l^a l_k + U_{ab} l^a l^b l_k l_h\right)_{|l|m} = a_{lm}\left(U_{kh} - U_{kb} l^b l_h - U_{ah} l^a l_k + U_{ab} l^a l^b l_k l_h\right).$$

Now, in view of (1.1) and if $U_{kb}\, y^b = 0 = U_{ah}\, y^a$, then above equation becomes

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$U_{kh|l|m} = a_{lm} U_{kh}$.

Thus, we conclude

Theorem 5.7. *In $U^h - BRF_n$, if the projection of the $U-$Ricci tensor U_{kh} on indicatrix behaves as birecurrent, then the $U-$Ricci tensor U_{kh} also behave as birecurrent, provided $U_{kb} y^b = 0 = U_{ah} y^a$.*

We know that the projection of the curvature tensor D^i_{jkh} on indicatrix behaves as birecurrent *i.e.* satisfied (5.8). Using (2.9) in (5.8), we get

$$\left(D^a_{bcd} h^i_a h^b_j h^c_k h^d_h\right)_{|l|m} = a_{lm} D^a_{bcd} h^i_a h^b_j h^c_k h^d_h .$$

Using (2.10) in above equation, we get

$$\left\{D^a_{bcd}\left(\delta^i_a - \ell^i \ell_a\right)\left(\delta^b_j - \ell^b \ell_j\right)\left(\delta^c_k - \ell^c \ell_k\right)\left(\delta^d_h - \ell^d \ell_h\right)\right\}_{|l|m}$$

$$= a_{lm}\left\{D^a_{bcd}\left(\delta^i_a - \ell^i \ell_a\right)\left(\delta^b_j - \ell^b \ell_j\right)\left(\delta^c_k - \ell^c \ell_k\right)\left(\delta^d_h - \ell^d \ell_h\right)\right\}$$

which can be written as

$$(D^i_{jkh} - D^i_{jkd}\ell^d \ell_h - D^i_{jch}\ell^c \ell_k + D^i_{jcd}\ell^c \ell_k \ell^d \ell_h - D^a_{jkh}\ell^i \ell_a$$

$$+ D^a_{jkd}\ell^i \ell_a \ell^d \ell_h + D^a_{jch}\ell^i \ell_a \ell^c \ell_k - D^a_{jcd}\ell^i \ell_a \ell^c \ell_k \ell^d \ell_h)_{|l|m}$$

$$= a_{lm}(D^i_{jkh} - D^i_{jkd}\ell^d \ell_h - D^i_{jch}\ell^c \ell_k + D^i_{jcd}\ell^c \ell_k \ell^d \ell_h - D^a_{jkh}\ell^i \ell_a$$

$$+ D^a_{jkd}\ell^i \ell_a \ell^d \ell_h + D^a_{jch}\ell^i \ell_a \ell^c \ell_k - D^a_{jcd}\ell^i \ell_a \ell^c \ell_k \ell^d \ell_h) .$$

Using(1.1) in above equation, then use (2.8), we get

(5.11) $$\left(D^i_{jkh} - D^a_{jkh}\ell^i \ell_a\right)_{|l|m} = a_{lm}\left(D^i_{jkh} - D^a_{jkh}\ell^i \ell_a\right).$$

Thus, we conclude

Theorem 5.8. *If the projection of the tensor $\left(D^i_{jkh} - D^a_{jkh}\ell^i \ell_a\right)$ on indicatrix is birecurrent, then the space is $U^h - BRF_n$.*

From (5.11), we get

Corollary 5.3. *In $U^h - BRF_n$, the projection of the tensor D^i_{jkh} on indicatrix is birecurrent, if and only if $D^a_{jkh}\ell^i \ell_a$ is birecurrent.*

VI. Conclusion

The relationship between some connection coefficients for different tensors was discussed. We studied different tensors which satisfy the birecurrence property. Also, we discussed the projection on indicatrix in sense of Cartan for some tensors which behaves as birecurrentin $U^h - BRF_n$.

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