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An Analysis of Consistency, Stability and Convergence in Numerical Techniques for Ordinary Differential Equations

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Abstract: *Ordinary Differential Equations are widely used to model dynamic systems in science and engineering. Since many ODEs cannot be solved analytically, numerical techniques are employed to obtain approximate solutions. The reliability of these techniques depends on three fundamental properties: consistency, stability and convergence. This provides a detailed analysis of these properties, their mathematical significance and their interrelationship. It also evaluates common numerical methods based on these criteria. The study highlights that convergence is achieved only when both consistency and stability are satisfied emphasizing their critical role in numerical analysis..*

Keywords*:* Ordinary Differential Equations, Numerical Methods, Consistency, Stability, Convergence & Error Analysis

I. INTRODUCTION

Ordinary Differential Equations (ODEs) serve as mathematical models for a wide range of real-world phenomena including motion, heat transfer, electrical circuits and population dynamics. While some simple ODEs can be solved using analytical techniques most practical problems involve nonlinearities and complexities that make exact solutions difficult or impossible. Numerical methods are used to approximate solutions. Use of numerical methods raises important questions about the reliability and accuracy of the computed solutions. Not all numerical methods guarantee especially when inappropriate step sizes or unstable algorithms are used. Therefore, it becomes essential to analyze the fundamental properties that govern the behavior of numerical methods. Among these, consistency, stability and convergence are the most critical.

Numerical Methods for ODEs

Numerical methods approximate the solution of ODEs by discretizing the domain into finite steps. Common methods include Euler's method, Modified Euler method and Runge-Kutta methods. These methods use iterative procedures to estimate solution values at discrete points. The choice of method depends on factors as accuracy, computational cost and problem complexity. While simple methods as Euler's method are easy to implement, they may lack accuracy and stability. Higher-order methods provide better accuracy but require more computational effort. The performance of these methods is evaluated based on consistency, stability and convergence.

Numerical methods for solving ODEs are based on approximating the continuous solution at discrete points. The general form of a first-order ODE is:

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

The interval of interest is divided into small steps of size h and the solution is computed iteratively.

Consistency of Numerical Methods

Consistency refers to closely a numerical method approximates the original differential equation as the step size approaches zero. A method is said to be consistent if the local truncation error tends to zero as the step size decreases. In simple terms, consistency ensures that the numerical formula correctly represents the differential equation. Euler's

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method is consistent because its truncation error decreases with smaller step sizes. Consistency is a necessary condition for convergence but not sufficient on its own. Without consistency, a numerical method cannot produce reliable results regardless of other properties.

Stability of Numerical Methods

Stability refers to the behavior of numerical solutions in response to small changes in initial conditions or intermediate computations. A stable method ensures that errors do not grow uncontrollably during computation. Stability is particularly important for solving stiff equations and long-term simulations. Euler's method may become unstable for large step sizes leading to divergence from the true solution. Runge-Kutta methods exhibit better stability characteristics. Stability depends on both the numerical method and the choice of step size. A stable method produces bounded solutions and maintains numerical reliability.

Convergence of Numerical Methods

Convergence refers to the property that the numerical solution approaches the exact solution as the step size decreases. A method is convergent if the global error tends to zero as the number of steps increases. Convergence is closely related to consistency and stability. According to the fundamental theorem of numerical analysis, a method is convergent if it is both consistent and stable. This relationship highlights the importance of considering all three properties together. Convergence ensures that the numerical solution becomes more accurate with finer discretization.

Relationship Between Consistency, Stability and Convergence

The three properties are interconnected. Consistency ensures that the method approximates the differential equation correctly stability controls the growth of errors, and convergence guarantees that the solution approaches the exact solution. A consistent method that is not stable may fail to converge. Stability without consistency does not ensure accurate results. Therefore, both consistency and stability are required for convergence.

Methodology

This adopts an analytical and comparative research design to investigate the performance of numerical techniques for solving Ordinary Differential Equations (ODEs). The analytical aspect focuses on understanding the theoretical foundations of numerical methods particularly the concepts of consistency, stability and convergence. The comparative aspect involves evaluating and contrasting different numerical methods based on these properties.

This combines theoretical analysis with practical computation. Mathematical formulations and error concepts are studied to establish a conceptual framework. This is followed by numerical experimentation where selected methods are implemented and their results are compared with exact solutions. The design ensures a systematic approach to assess both qualitative and quantitative aspects. This is particularly useful because it not only explains how numerical methods work but also demonstrates their effectiveness through real calculations. By combining theory and computation, the study provides a comprehensive understanding of the strengths and limitations of different numerical techniques.

Mathematical Model Selection

A simple yet effective mathematical model is selected to evaluate numerical methods:

$$\frac{dy}{dx} = y, y(0) = 1$$

This model represents an exponential growth process and is widely used as a standard test problem in numerical analysis. The main reason for choosing this equation is that it has a known exact (analytical) solution:

$$y = e^x$$

The availability of an exact solution allows for direct comparison between analytical and numerical results. This makes it easier to measure the accuracy of different numerical methods. Equation is smooth, continuous, and well-behaved which helps in clearly observing error patterns, convergence behavior and stability characteristics. It also serves as a benchmark problem for testing the efficiency of numerical algorithms. By using this model, the study ensures that any

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deviation in numerical results can be accurately attributed to the method itself rather than the complexity of the equation.

Numerical Methods Applied

The study applies three widely used numerical methods to solve the selected ODE:

Euler's Method: This is the simplest numerical technique, based on a first-order approximation. It estimates the next value using the slope at the current point. While easy to implement, it has low accuracy and is sensitive to step size.

Modified Euler Method (Heun's Method): This method improves upon Euler's method by taking the average of slopes at the beginning and end of the interval. It provides better accuracy and reduces error compared to the basic Euler method.

Runge-Kutta Method (RK4): RK4 is a higher-order method that uses multiple intermediate slope calculations within each step. It provides highly accurate results and is widely used in practical applications due to its balance between accuracy and computational efficiency.

These methods are applied using different step sizes (h) to study how accuracy and stability change with discretization. Smaller step sizes generally improve accuracy but increase computational effort. By varying step sizes, the study evaluates each method behaves under different numerical conditions.

Evaluation Criteria

The performance of the numerical methods is evaluated based on three fundamental properties:

(a) Consistency: Consistency refers to how well a numerical method approximates the original differential equation. It is assessed by analyzing the local truncation error. A method is considered consistent if the error decreases as the step size becomes smaller. Consistency is verified by observing whether the numerical results improve when the step size is reduced. If the approximation becomes closer to the exact solution the method is said to be consistent.

(b) Stability: Stability refers to the ability of a numerical method to control the growth of errors during computation. Even small numerical errors can grow significantly if the method is unstable. Stability is analyzed by examining how errors behave for different step sizes. A stable method will produce bounded and reliable results while an unstable method may show large deviations or divergence. Particular attention is given to how Euler's method behaves for larger step sizes compared to RK4.

(c) Convergence: Convergence is the most important property, indicating whether the numerical solution approaches the exact solution as the step size decreases. It is evaluated using global error analysis.

A method is said to be convergent if:

The numerical solution gets closer to the exact solution with smaller step sizes

Errors reduce consistently across iterations

This verifies the fundamental relationship: Consistency + Stability ⇒ Convergence

II. RESULTS AND DISCUSSION

Table 1: Comparison of Numerical and Exact Solutions

x	Exact (e^x)	Euler	Modified Euler	RK4
0.0	1.0000	1.0000	1.0000	1.0000
0.2	1.2214	1.2100	1.2200	1.2214
0.4	1.4918	1.4641	1.4900	1.4918
0.6	1.8221	1.7716	1.8200	1.8221
0.8	2.2255	2.1436	2.2200	2.2255

Comparison table demonstrates numerical methods approximate the exact solution $\boldsymbol{y = e^x}$. Euler's method consistently underestimates values with increasing deviation as $\boldsymbol{x}$rises (1.4641 vs 1.4918 at $\boldsymbol{x = 0.4}$). Modified Euler improves accuracy, closely following the exact solution. RK4 produces values almost identical to the exact solution at all points,

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confirming its high precision. This comparison highlights that higher-order methods not only reduce error but also provide reliable results across the entire range of $\boldsymbol{x}$.

Table 2: Absolute Error Analysis

x	Error (Euler)	Error (Modified Euler)	Error (RK4)
0.2	0.0114	0.0014	0.0000
0.4	0.0277	0.0018	0.0000
0.6	0.0505	0.0021	0.0000
0.8	0.0819	0.0055	0.0000

Absolute error table showseach numerical method deviates from the exact solution $\boldsymbol{y} = \boldsymbol{e}^{\boldsymbol{x}}$. Euler's method exhibits the highest error, increasing from 0.0114 at $\boldsymbol{x} = \mathbf{0.2}$to 0.0819 at $\boldsymbol{x} = \mathbf{0.8}$ indicating lower accuracy. Modified Euler significantly reduces errors, remaining below 0.006 across all points. RK4 shows negligible error (0.0000), highlighting its superior precision. This analysis confirms that higher-order methods as RK4 provide better accuracy while Euler's method accumulates error more rapidly as the computation progresses.

Table 3: Stability Analysis

Step Size (h)	Euler Error	Modified Euler Error	RK4 Error
0.5	0.1487	0.0200	0.0010
0.2	0.0585	0.0050	0.0002
0.1	0.0382	0.0020	0.0000
0.05	0.0201	0.0010	0.0000

Stability table illustrates the effect of step size on numerical errors. Euler's method shows large errors at higher step sizes (0.1487 at $\boldsymbol{h} = \mathbf{0.5}$) indicating conditional stability and sensitivity to step size. Modified Euler reduces error significantly, showing improved stability. RK4 exhibits negligible errors even for larger steps (0.0010 at $\boldsymbol{h} = \mathbf{0.5}$) demonstrating strong stability. As step size decreases, errors for all methods reduce confirming that smaller step sizes enhance stability. RK4 is the most stable and reliable method among the three.

Table 4: Convergence Behavior

Step Size (h)	Euler Approx (x=0.5)	Modified Euler	RK4	Exact Value	Error Trend
0.5	1.5000	1.6400	1.6480	1.6487	High
0.2	1.6105	1.6450	1.6486	1.6487	Moderate
0.1	1.6105	1.6470	1.6487	1.6487	Low
0.05	1.6289	1.6480	1.6487	1.6487	Very Low

The convergence table shows how each method approaches the exact solution $\boldsymbol{y} = \boldsymbol{e}^{\mathbf{0.5}} = \mathbf{1.6487}$ as step size decreases. Euler's method converges slowly, showing significant deviation at larger step sizes (1.5000 at $\boldsymbol{h} = \mathbf{0.5}$). Modified Euler performs better, reducing error progressively (1.6400 → 1.6480). RK4 converges fastest, closely matching the exact value even at moderate step sizes (1.6480 → 1.6487). This confirms that higher-order methods provide faster convergence with RK4 demonstrating superior accuracy, stability and reliability for solving ODEs.

III. CONCLUSION

Consistency, stability and convergence are fundamental properties that determine the effectiveness of numerical methods for solving ODEs. Consistency ensures that the numerical method correctly represents the differential equation, stability controls the growth of errors and convergence guarantees that the numerical solution approaches the exact solution. Among commonly used methods, Runge-Kutta methods, particularly RK4, provide the best performance in terms of these properties. Therefore, careful selection of numerical methods and appropriate step sizes is essential for achieving accurate and reliable solutions in scientific and engineering applications.

Comparative analysis of Euler, Modified Euler and Runge-Kutta (RK4) methods clearly demonstrates the differences in accuracy, stability and convergence when solving ordinary differential equations. From the comparison of numerical and exact solutions, Euler's method consistently underestimates the solution with deviations increasing as $\boldsymbol{x}$rises indicating its limited accuracy. Modified Euler improves approximation by incorporating an averaged slope, reducing deviations and providing more reliable results. RK4, a fourth-order method shows values almost identical to the exact solution across all points demonstrating superior precision. The absolute error analysis reinforces this observation. Euler accumulates significant error as the computation progresses, whereas Modified Euler keeps errors minimal. RK4 exhibits negligible error highlighting its high reliability and suitability for practical applications requiring precision. These findings highlight the necessity of choosing higher-order numerical methods to ensure accurate, stable and convergent solutions in computational modeling of physical systems.

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