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Generalized Tri-recurrent Finsler Space Under Cartan-Type Mixed Covariant Derivatives

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Abstract: *In this paper, we concentrate on a new Finsler space that is a combination of two types of Cartan derivatives of third order for Berwald curvature tensor H^i_{jkh}. We find the condition that Berwald curvature tensor H^i_{jkh} is special generalized of generalized mixed trirecurrent. Furthermore, we show that the normal projective curvature tensor N^i_{jkh} is generalized $hv-$ mixed trirecurrent if and only if the tensor $\dot{\partial}_j(H_{hk}-H_{kh})$ behaves as mixed trirecurrent. Also, the Ricci tensor N_{jk} of the normal projective curvature tensor N^i_{jkh} is non – vanishing if and only if the tensor $((1-n)\dot{\partial}_j\dot{\partial}_k H + H_{jk} + H_{kj})$ is mixed trirecurrent.*

Keywords: Generalized H^{hv} – mixed trirecurrent space, $hv-$covariant derivative of mixed second order, Berwald curvature tensor H^i_{jkh} , Normal curvature tensor N^i_{jkh}

I. INTRODUCTION

Several curvature tensors have been shown to satisfy the generalized birecurrence condition through the use of two different types of Cartan covariant derivatives, as introduced by [3, 7, 12]. The generalized birecurrent Finsler space of mixed covariant derivatives in Cartan sense for various tensor introduced by Al – Qashbari et al. [4, 5]. The relationship between two curvature tensors in Finsler spaces have been discussed by [1]. Some tensors which satisfy the trirecurrence property in Cartan sense studied by [6, 11].

Let F_n be an $n-$dimensional Finsler space equipped with the metric function $F(x,y)$ satisfying the request conditions [8, 14]. The vectors y_i and y^i defined by

(1.1) a) $y_i = g_{ij}(x,y)y^j$ and b) $\dot{\partial}_j\, y_i = g_{ij}$.

The metric tensor g_{ij} and its associative g^{ij} are connected by

(1.2) $g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j=k\,, \\ 0 & if\ \ j\neq k\,. \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) a) $\delta^i_j y_i = y_j$, b) $\delta^i_j\, y^j = y^i$, c) $\delta^i_j g_{ir} = g_{jr}$,

d) $\delta^i_j\, g^{jk} = g^{ik}$, e) $y_i\, y^i = F^2$ and f) $\delta^i_i = n$.

The $(h)hv-$ torsion tensor which is positively homogeneous of degree -1 in y^i and defined by [9, 10, 15]

$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2$.

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and b) $C^h_{jk}g_{ih} = C_{ijk}$.

Cartan [7] deduced the covariant derivatives of an arbitrary vector field X^i with respect to x^k which given by

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(1.5) $X^i\big|_k = \dot{\partial}_k X^i + X^r C^{\ i}_{rk}$

(1.6) $X^i_{|k} = \partial_k X^i - \left(\dot{\partial}_r X^i\right) G^{\ r}_k + X^r \Gamma^{*\ i}_{rk}$,

where the function Γ^{*i}_{rk} is defined by $\Gamma^{*i}_{rk} = \Gamma^{\ i}_{rk} - C^{\ i}_{mr}\, \Gamma^{\ m}_{sk}\ y^s$. The functions Γ^{*i}_{rk} and G^r_k are connected by $G^{\ r}_k = \Gamma^{*r}_{sk}\, y^s$ where $\partial_j \equiv \frac{\partial}{\partial x^j}$, $\dot{\partial}_j \equiv \frac{\partial}{\partial y^j}$, $G^i_j = \dot{\partial}_j G^i$.

The equations (1.5) and (1.6) give two kinds of covariant differentiations which are called v −covariant differentiation and h −covariant differentiation, respectively. So $X^i|_k$ and $X^i_{|k}$ are v −covariant derivative and h −covariant derivative of the vector field X^i.

Therefore, v −covariant derivative and h −covariant derivative of the vectors y^i , y_i and metric tensors g_{ij} and its associative g^{ij} are satisfied [14]

(1.7) a) $g_{ij}\big|_k = 0$, b) $g_{ij|k} = 0$, c) $g^{ij}\big|_k = 0$, d) $g^{ij}_{|k} = 0$,

e) $y^i_{|k} = 0$, f) $y^i\big|_k = \delta^i_k$, g) $y_{j|k} = 0$ and h) $y_j|_k = g_{jk}$.

For an arbitrary vector field x^i . the two processes of covariant differentiation, defined above commute with partial differentiation with respect to y^i according to

(1.8) $\dot{\partial}_j\left(X^i\big|_k\right) - \left(\dot{\partial}_j X^i\right)\big|_k = X^h\left(\dot{\partial}_j C^i_{kh}\right) + C^h_{kj}\left(\dot{\partial}_h X^i\right)$.

(1.9) $\dot{\partial}_j\left(X^i_{|k}\right) - \left(\dot{\partial}_j X^i\right)_{|k} = X^r\left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk}$,

where

(1.10) $P^r_{jk} = \left(\dot{\partial}_k \Gamma^{*r}_{hj}\right) y^h = \Gamma^{*r}_{hjk} y^h$.

The Berwald curvature tensor $H^{\ i}_{jkh}$ is defined by [2]

$$H^{\ i}_{jkh} = \partial_h G^{\ i}_{jk} + G^{\ s}_{jk} G^{\ i}_{sh} + G^{\ i}_{sjh} G^{\ s}_{k} - h/k$$

and satisfied

(1.11) a) $H^{\ i}_{jkh}\, y^j = H^{\ i}_{kh}$, b) $H^i_{jk}\, y^j = H^{\ i}_k$, c) $H^{\ i}_{jki} = H_{jk}$, d) $H^{\ i}_{ki} = H_k$ e) $H^{\ i}_i = (n-1)\, H$,

f) $\dot{\partial}_j H^{\ i}_{kh} = H^{\ i}_{jkh}$, g) $H_{jk} = \dot{\partial}_j H_k$, h) $H_{jk} y^{\ j} = H_k$ i) $H_{jk} y^{\ k} = (n-1)\dot{\partial}_j H - H_j$, j) $H_k y^{\ k} = (n-1)H$, k) $H_{ijkh} = g_{jr}\, H^{\ r}_{ikh}$

And l) $H^{\ r}_{rkh} = H_{hk} - H_{kh}$,

where H_{ijkh} , $H^{\ i}_{kh}$, $H^{\ i}_k$, H_{jk} , H_k and H are the associate curvature tensor, torsion tensor, deviation tensor, H −Ricci tensor, curvature vector and scalar curvature of Berwald curvature tensor $H^{\ i}_{jkh}$.

Qasem and Hadi [13] introduced Finsler space F_n which Berwald curvature tensor H^i_{jkh} satisfies the generalized birecurrence property with respect to Cartan's second kind covariant derivative, i.e. characterized by the condition [13]

(1.12) $H^i_{jkh|l|m} = a_{lm} H^i_{jkh} + b_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $H^i_{jkh} \neq 0$,

where a_{lm} and b_{lm} are non- zero covariant tensor fields of second order. This space called a *generalized H^h −birecurrent space* and denoted it briefly by $GH^h - BRF_n$.

Transvecting the condition (1.12) by y^j, using (1.11), (1.7) and (1.1), we get

(1.13) $H^i_{kh|l|m} = a_{lm}H^i_{kh} + b_{lm}(\delta^i_k y_h - \delta^i_h y_k)$.

II. A GENERALIZED H^{hv} – MIXED TRIRECURRENT SPACE

This section introduces a Finsler space which Berwald curvature tensor H^i_{jkh} satisfies the generalized trirecurrence property by using two kinds of covariant derivatives in Cartan sense.

Let us consider that the Berwald curvature tensor H^i_{jkh} satisfies the special generalized recurrence property with respect to Cartan's first kind covariant derivative, i.e. characterized by the condition

(2.1) $H^i_{jkh}|_n = \lambda_n H^i_{jkh}$ $\qquad H^i_{jkh} \neq 0$,

where $|n$ is the $v-$covariant derivative operator of first order with respect to x^n. A Finsler space F_n which H^i_{jkh} satisfies the condition (2.1) will be called a H^v −*recurrent space* and denote it briefly by $H^v - RF_n$.

Now, taking v – covariant derivative for (1.12) and using (1.7) with respect to x^n, we get

$$H^i_{jkh|l|m|n} = a_{lm|n}H^i_{jkh} + a_{lm}H^i_{jkh|n} + b_{lm|n}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$

Using the condition (2.1) in above condition, we get

$$H^i_{jkh|l|m|n} = a_{lm}|_n H^i_{jkh} + a_{lm}[\lambda_n H^i_{jkh}] + b_{lm|n}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$

Or

$$H^i_{jkh|l|m|n} = (a_{lm|n} + a_{lm}\lambda_n)H^i_{jkh} + b_{lm|n}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

The above equation can be written as

(2.2) $H^i_{jkh|l|m|n} = c_{lmn}H^i_{jkh} + d_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$. , $\quad R^i_{jkh} \neq 0$,

where $|l|m|n$ is hv −covariant differential operator of third mixed order with respect to x^l ,x^m and x^n, respectively. Also, $c_{lmn} = a_{lm|n} + a_{lm}\lambda_n$ and $d_{lmn} = b_{lm|n}$ are non − zero covariant tensors field of third order.

Definition 2.1. *If Berwald curvature tensor H^i_{jkh} of a Finsler space satisfying the condition (2.2), where c_{lmn} and d_{lmn} are non-zero covariant tensor fields of third order, we denote such space a generalized H^{hv} – mixed trirecurrent space and a tensor will be called a generalized hv – mixed trirecurrent tensor. This space and tensor denote them briefly by $GH^{hv} - (M)TRF_n$ and $Ghv - (M)TR$, respectively.*

Now, transvecting the condition (2.2) by g_{ip}, using (1.11), (1.3) and (1.7), we get

(2.3) $H_{jpkh|l|m|n} = c_{lmn}H_{jpkh} + d_{lmn}(g_{kp}g_{jh} - g_{hp}g_{jk})$.

Remark 2.1. Conversely, transvecting (2.3) by the associate metric tensor g^{ip} and using (1.11), (1.7) and (1.2), yields the condition (2.2), i.e. the condition (2.2) is equivalent (2.3), therefore $GH^{hv} - (M)TRF_n$ can represent by the condition (2.3). Thus, we conclude

Corollary 2.1. *In $GH^{hv} - (M)TRF_n$ may characterize by (2.3).*

Transvecting (2.2) by y^j , using (1.11) and (1.1), we get

$$H^i_{kh|l|m|n} = H^i_{jkh|l|m|n}y^j + H^i_{jkh|l|m}\, y^j_{|n} + c_{lmn}H^i_{kh} + d_{lmn}(\delta^i_k y_h - \delta^i_h y_k).$$

Using the condition (1.12) and (2.2) in the above equation, using (1.3), (1.7) and (1.7), we get

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$$H^i_{kh|l|m|n} = c_{lmn}H^i_{kh} + d_{lmn}(\delta^i_k y_h - \delta^i_h y_k) + a_{lm}H^i_{jkh}\delta^j_n$$
$$+b_{lm}(\delta^i_k g_{nh} - \delta^i_h g_{nk}) + c_{lmn}H^i_{kh} + d_{lmn}(\delta^i_k y_h - \delta^i_h y_k).$$

Or

(2.4) $H^i_{kh|l|m|n} = c_{lmn}H^i_{kh} + d_{lmn}(\delta^i_k y_h - \delta^i_h y_k) + a_{lm}H^i_{nkh} + b_{lm}(\delta^i_k g_{nh} - \delta^i_h g_{nk}),$

where $c_{lmn} = 2c_{lmn}$ and $d_{lmn} = 2d_{lmn}$.

Contracting the indies i and h in the condition (2.2), using (1.11) and (1.3), we get

(2.5) $H_{jk|l|m|n} = c_{lmn}H_{jk} + d_{lmn}(1-n)g_{jk}.$

Transvecting (2.4) by y^k using (1.7), we get

$$H^i_{h|l|m|n} = H^i_{kh|l|m|n}\, y^k + H^i_{kh|l|m} y^k_{|n} + c_{lmn}H^i_{kh}y^k + d_{lmn}(\delta^i_k y_h - \delta^i_h y_k)y^k$$
$$+a_{lm}H^i_{nkh}y^k + b_{lm}(\delta^i_k g_{nh} - \delta^i_h g_{nk})y^k.$$

Using the conditions (2.4) and (1.13) in the above equation, using (1.11) and (1.3), we get

$$H^i_{h|l|m|n} = c_{lmn}H^i_h + d_{lmn}(y^i y_h - \delta^i_h F^2) + a_{lm}H^i_{nh} + b_{lm}(y^i g_{nh} - \delta^i_h y_n)$$
$$+a_{lm}H^i_{nh} + b_{lm}(\delta^i_n y_h - \delta^i_h y_n) + c_{lmn}H^i_h + d_{lmn}(y^i y_h - \delta^i_h F^2) + a_{lm}H^i_{nkh}$$
$$+b_{lm}(y^i g_{nh} - \delta^i_h y_n).$$

Or

(2.6) $$H^i_{h|l|m|n} = c_{lmn}H^i_h + d_{lmn}(y^i y_h - \delta^i_h F^2) + a_{lm}H^i_{nh}$$
$$+b_{lm}(\delta^i_n y_h + 2y^i g_{nh} - 3\,\delta^i_h y_n).$$

Contracting the indies i and h in the condition (2.4), using (1.11) and (1.3), we get

(2.7) $H_{k|l|m|n} = c_{lmn}H_k + d_{lmn}(1-n)y_k + a_{lm}H_{nk} + b_{lm}(1-n)g_{nk}.$

Contracting the indies i and h in the condition (2.6), using (1.11), (1.3) and (1.1), we get

(2.8) $H_{|l|m|n} = c_{lmn}H + d_{lmn}(1-n)F^2 + a_{lm}H_n + 2b_{lm}(ny_h + 2y_n - 3ny_n).$

Thus, we conclude

Theorem 2.1. *In $GH^{hv} - (M)TRF_n$, the torsion tensor H^i_{kh} , Ricci tensor H_{jk}, deviation tensor H^i_h, curvature vector H_k and curvature scalar H are giving by (2.4), (2.5), (2.6), (2.7) and (2.8), respectively.*

III. NECESSARY AND SUFFICIENT CONDITION FOR SOME TENSORS IN $GH^{hv} - (M)TRF_n$

Differentiating the condition (2.4) partially with respect to y^j, using (1.1) and (1.4), we get

(3.1) $$\dot\partial_j(H^i_{kh|l|m|n}) = (\dot\partial_j c_{lmn})H^i_{kh} + c_{lmn}(\dot\partial_j H^i_{kh}) + (\dot\partial_j d_{lmn})(\delta^i_k y_h - \delta^i_h y_k)$$
$$+d_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + (\dot\partial_j a_{lm})H^i_{nkh} + a_{lm}(\dot\partial_j H^i_{nkh})$$
$$+(\dot\partial_j b_{lm})(\delta^i_k g_{nh} - \delta^i_h g_{nk}) + 2b_{lm}(\delta^i_k C_{hnj} - \delta^i_h C_{knj}).$$

Using the commutation formula exhibited by (1.8) for the $h(v)$- torsion tensor $H^i_{kh|l|m}$ in the condition (2.4), we get

(3.2) $(\dot\partial_j H^i_{kh|l|m})_{|n} + H^r_{kh|l|m}(\dot\partial_j C^i_{nr}) - H^i_{rh|l|m}(\dot\partial_j C^r_{kn}) - H^i_{kr|l|m}(\dot\partial_j C^r_{hn})$

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$$-H^i_{kh|l|m}(\dot\partial_j C^r_{ln}) - H^i_{kh|l|r}(\dot\partial_j C^r_{mn}) + C^r_{nj}(\dot\partial_r H^i_{kh|l|m}) = (\dot\partial_j c_{lmn})H^i_{kh}$$
$$+c_{lmn}H^i_{jkh} + (\dot\partial_j d_{lmn})(\delta^i_k y_h - \delta^i_h y_k) + d_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + (\dot\partial_j a_{lm})H^i_{nkh}$$
$$+a_{lm}(\dot\partial_j H^i_{nkh}) + (\dot\partial_j b_{lm})(\delta^i_k g_{nh} - \delta^i_h g_{nk}) + 2b_{lm}(\delta^i_k C_{hnj} - \delta^i_h C_{knj}).$$

Applying the commutative formula (1.9) for $H^i_{kh|l}$ in (3.2), we get

(3.3)
$$[\dot\partial_j (H^i_{kh|l})_{|m} + H^r_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm})$$
$$-(\dot\partial_r H^i_{kh|l})P^r_{jm}]_{|n} + H^r_{kh|l|m}(\dot\partial_j C^i_{nr}) - H^i_{rh|l|m}(\dot\partial_j C^r_{kn}) - H^i_{kr|l|m}(\dot\partial_j C^r_{hn})$$
$$-H^i_{kh|r|m}(\dot\partial_j C^r_{ln}) - H^i_{kh|l|r}(\dot\partial_j C^r_{mn}) + C^r_{nj}(\dot\partial_r H^i_{kh|l|m}) = (\dot\partial_j c_{lmn})H^i_{kh}$$
$$+c_{lmn}H^i_{jkh} + (\dot\partial_j d_{lmn})(\delta^i_k y_h - \delta^i_h y_k) + d_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + (\dot\partial_j a_{lm})H^i_{nkh}$$
$$+a_{lm}(\dot\partial_j H^i_{nkh}) + (\dot\partial_j b_{lm})(\delta^i_k g_{nh} - \delta^i_h g_{nk}) + 2b_{lm}(\delta^i_k C_{hnj} - \delta^i_h C_{knj}).$$

Again, applying the commutative formula (1.9) for H^i_{kh}in (3.3), using (1.11), we get

(3.5)
$$H^i_{jkh|l|m|n} + [H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rl})]_{|m|n} - [H^i_{rh}(\dot\partial_j \Gamma^{*r}_{kl})]_{|m|n} - [(\dot\partial_r H^i_{kh})P^r_{jl}]_{|m|n}$$
$$+[H^r_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm})]_{|n} - [H^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km})]_{|n} - [H^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm})]_{|n} - [H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm})]_{|n}$$
$$-[(\dot\partial_r H^i_{kh|l})P^r_{jm}]_{|n} + H^r_{kh|l|m}(\dot\partial_j C^i_{nr}) - H^i_{rh|l|m}(\dot\partial_j C^r_{kn}) - H^i_{kr|l|m}(\dot\partial_j C^r_{hn})$$
$$-H^i_{kh|r|m}(\dot\partial_j C^r_{ln}) - H^i_{kh|l|r}(\dot\partial_j C^r_{mn}) + C^r_{nj}(\dot\partial_r H^i_{kh|l|m}) = (\dot\partial_j c_{lmn})H^i_{kh}$$
$$+c_{lmn}H^i_{jkh} + (\dot\partial_j d_{lmn})(\delta^i_k y_h - \delta^i_h y_k) + d_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + (\dot\partial_j a_{lm})H^i_{nkh}$$
$$+a_{lm}(\dot\partial_j H^i_{nkh}) + (\dot\partial_j b_{lm})(\delta^i_k g_{nh} - \delta^i_h g_{nk}) + 2b_{lm}(\delta^i_k C_{hnj} - \delta^i_h C_{knj}).$$

This show that

(3.6)
$$H^i_{jkh|l|m|n} = c_{lmn}H^i_{jkh}$$

if and only if

(3.7)
$$[H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rl})]_{|m|n} - [H^i_{rh}(\dot\partial_j \Gamma^{*r}_{kl})]_{|m|n} - [(\dot\partial_r H^i_{kh})P^r_{jl}]_{|m|n}$$
$$+[H^r_{kh|l}(\dot\partial_j \Gamma^{*i}_{rm})]_{|n} - [H^i_{rh|l}(\dot\partial_j \Gamma^{*r}_{km})]_{|n} - [H^i_{kr|l}(\dot\partial_j \Gamma^{*r}_{hm})]_{|n} - [H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{lm})]_{|n}$$
$$-[(\dot\partial_r H^i_{kh|l})P^r_{jm}]_{|n} + H^r_{kh|l|m}(\dot\partial_j C^i_{nr}) - H^i_{rh|l|m}(\dot\partial_j C^r_{kn}) - H^i_{kr|l|m}(\dot\partial_j C^r_{hn})$$
$$-H^i_{kh|r|m}(\dot\partial_j C^r_{ln}) - H^i_{kh|l|r}(\dot\partial_j C^r_{mn}) + C^r_{nj}(\dot\partial_r H^i_{kh|l|m}) = (\dot\partial_j c_{lmn})H^i_{kh}$$
$$+(\dot\partial_j d_{lmn})(\delta^i_k y_h - \delta^i_h y_k) + d_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + (\dot\partial_j a_{lm})H^i_{nkh}$$
$$+a_{lm}(\dot\partial_j H^i_{nkh}) + (\dot\partial_j b_{lm})(\delta^i_k g_{nh} - \delta^i_h g_{nk}) + 2b_{lm}(\delta^i_k C_{hnj} - \delta^i_h C_{knj}).$$

Thus, we conclude

Theorem 3.1. *In $GH^{hv} - (M)TRF_n$, the Berwald curvature tensor H^i_{jkh} is special generalized of generalized H^{hv} −mixed trirecurrent if and only if (3.7) holds.*

Differentiating the condition (2.7) partially with respect to y^j, using (1.1) and (1.4), we get

(3.8)
$$\dot\partial_j \left(H_{k|l|m|n}\right) = \dot\partial_j(c_{lmn})H_k + c_{lmn}(\dot\partial_j H_k) + (\dot\partial_j d_{lmn})(1-n)y_k$$
$$+d_{lmn}(1-n)g_{jk} + (\dot\partial_j a_{lm})H_{nk} + a_{lm}(\dot\partial_j H_{nk}) + (\dot\partial_j b_{lm})(1-n)g_{nk}$$

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$+2b_{lm}(1-n)C_{jnk}.$

Using the commutation formula exhibited by (1.8) for the curvature vector $H_{k|l|m}$ in (2.7), we get

(3.9) $\left(\dot{\partial}_j H_{k|l|m}\right)_{|n} - H_{r|l|m}\left(\dot{\partial}_j C^r_{kn}\right) - H_{k|r|m}\left(\dot{\partial}_j C^r_{ln}\right) - H_{k|l|r}\left(\dot{\partial}_j C^r_{mn}\right)$

$+C^r_{nj}\left(\dot{\partial}_r H_{k|l|m}\right) = \left(\dot{\partial}_j\, c_{lmn}\right)H_k + c_{lmn}\left(\dot{\partial}_j H_k\right) + \left(\dot{\partial}_j d_{lmn}\right)(1-n)y_k$

$+d_{lmn}(1-n)g_{jk} + \left(\dot{\partial}_j a_{lm}\right)H_{nk} + a_{lm}(\,\dot{\partial}_j\, H_{nk}) + \left(\dot{\partial}_j b_{lm}\right)(1-n)g_{nk}$

$+2b_{lm}(1-n)C_{jnk}.$

Applying the commutative formula (1.9) for $H_{k|l}$ in (3.9), we get

(3.10) $\left[\left(\dot{\partial}_j H_{k|l}\right)_{|m} - H_{r|l}\left(\dot{\partial}_j \Gamma^{*r}_{km}\right) - H_{k|r}\left(\dot{\partial}_j \Gamma^{*r}_{lm}\right) - \left(\dot{\partial}_r H_{k|l}\,\right)P^r_{jm}\right]_{|n} - H_{r|l|m}\left(\dot{\partial}_j C^r_{kn}\right)$

$- H_{k|r|m}\left(\dot{\partial}_j C^r_{ln}\right) - H_{k|l|r}\left(\dot{\partial}_j C^r_{mn}\right) + C^r_{nj}\left(\dot{\partial}_r H_{k|l|m}\right) = \left(\dot{\partial}_j\, c_{lmn}\right)H_k$

$+c_{lmn}\left(\dot{\partial}_j H_k\right) + \left(\dot{\partial}_j d_{lmn}\right)(1-n)y_k + d_{lmn}(1-n)g_{jk} + \left(\dot{\partial}_j a_{lm}\right)H_{nk}$

$+a_{lm}(\,\dot{\partial}_j\, H_{nk}) + \left(\dot{\partial}_j b_{lm}\right)(1-n)g_{nk} + 2b_{lm}(1-n)C_{jnk}.$

Again, applying the commutative formula (1.9) for H_k in (3.9), using (1.11), we get

(3.11) $\left\{\left[\left(H_{jk}\right)_{|l} - H_r\left(\dot{\partial}_j \Gamma^{*r}_{kl}\right) - \left(H_{rk}\,\right)P^r_{jl}\right]_{|m} - H_{r|l}\left(\dot{\partial}_j \Gamma^{*r}_{km}\right) - H_{k|r}\left(\dot{\partial}_j \Gamma^{*r}_{lm}\right) - \left(\dot{\partial}_r H_{k|l}\,\right)P^r_{jm}\right\}_{|n} - H_{r|l|m}\left(\dot{\partial}_j C^r_{kn}\right) -$

$H_{k|r|m}\left(\dot{\partial}_j C^r_{ln}\right) - H_{k|l|r}\left(\dot{\partial}_j C^r_{mn}\right)$

$+C^r_{nj}\left(\dot{\partial}_r H_{k|l|m}\right) = \dot{\partial}_j(\,c_{lmn})H_k + c_{lmn}\left(H_{jk}\right) + \dot{\partial}_j(d_{lmn})(1-n)y_k$

$+d_{lmn}(1-n)g_{jk} + \dot{\partial}_j(a_{lm})H_{nk} + a_{lm}(\,\dot{\partial}_j\, H_{nk}) + \dot{\partial}_j(b_{lm})(1-n)g_{nk}$

$+2b_{lm}(1-n)C_{jnk}.$

This show that

(3.12) $H_{jk|l|m|n} = c_{lmn}H_{jk}$

if and only if

(3.13) $\left\{\left[H_r\left(\dot{\partial}_j \Gamma^{*r}_{kl}\right) + \left(\dot{\partial}_r H_k\,\right)P^r_{jl}\right]_{|m} + H_{r|l}\left(\dot{\partial}_j \Gamma^{*r}_{km}\right) + H_{k|r}\left(\dot{\partial}_j \Gamma^{*r}_{lm}\right) + \left(\dot{\partial}_r H_{k|l}\,\right)P^r_{jm}\right\}_{|n} + H_{r|l|m}\left(\dot{\partial}_j C^r_{kn}\right) +$

$H_{k|r|m}\left(\dot{\partial}_j C^r_{ln}\right) + H_{k|l|r}\left(\dot{\partial}_j C^r_{mn}\right)$

$-C^r_{nj}(\dot{\partial}_r H_{k|l|m} + \dot{\partial}_j(\,c_{lmn})H_k + c_{lmn}\left(\dot{\partial}_j H_k\right) + \dot{\partial}_j(d_{lmn})(1-n)y_k$

$+d_{lmn}(1-n)g_{jk} + \dot{\partial}_j(a_{lm})H_{nk} + a_{lm}(\,\dot{\partial}_j\, H_{nk}) + \dot{\partial}_j(b_{lm})(1-n)g_{nk}$

$+2b_{lm}(1-n)C_{jnk} = 0.$

Thus, we conclude

Theorem 3.2. *In $GH^{hv}-(M)TRF_n$, the Ricci tensor H_{jk} behaves special generalized of generalized H^{hv} −mixed trirecurrent if and only if (3.13) holds.*

Pandey [11] and Yano [15] obtained the relation between the normal projective curvature tensor N^i_{jkh} that defined by

(3.14) $N^i_{jkh} = H^i_{jkh} - \frac{y^i}{n+1}\,\dot{\partial}_j H^r_{rkh}$

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Using (1.11) in above equation, then taking $hv-$ covarint derivative of mixed third order for (3.14) with respect to x^l, x^m and x^n, successively, we get

$$N^i_{jkh|l|m|n} = H^i_{jkh|l|m|n} - \frac{y^i}{n+1}\,[\dot\partial_j(H_{hk}-H_{kh})\,]_{|l|m|n}.$$

Using the condition (2.2) in above equation, then using (3.14), we get

(3.15) $$N^i_{jkh|l|m|n} = c_{lmn}N^i_{jkh} + c_{lmn}\frac{y^i}{n+1}\,\dot\partial_j(H_{hk}-H_{kh}) + d_{lmn}\,(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - \frac{y^i}{n+1}\,[\dot\partial_j(H_{hk}-H_{kh})\,]_{|l|m|n}.$$

This show that

(3.16) $$N^i_{jkh|l|m|n} = c_{lmn}N^i_{jkh} + d_{lmn}\,(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$

if and only if

(3.17) $$\frac{y^i}{n+1}\,[\dot\partial_j(H_{hk}-H_{kh})\,]_{|l|m|n} = c_{lmn}\frac{y^i}{n+1}\,\dot\partial_j(H_{hk}-H_{kh}).$$

Theorem 3.3. In $GH^{hv}-(M)TRF_n$, *the normal projective curvature tensor* N^i_{jkh} *is generalized* $hv-$ *mixed trirecurrent if and only if the tensor* $\dot\partial_j(H_{hk}-H_{kh})$ *behaves as mixed trirecurrent.*

Contracting the indices i and h in (3.15) and using (1.3), we get

$$N_{jk|l|m|n} = c_{lmn}N_{jk} + c_{lmn}\frac{y^i}{n+1}\,\dot\partial_j(H_{ik}-H_{ki}) + d_{lmn}\,(1-n)g_{jk} - \frac{y^i}{n+1}\,[\dot\partial_j(H_{ik}-H_{ki})\,]_{|l|m|n}.$$

Where $N^i_{jki} = N_{jk}$ the above equation can be written as

(3.18) $$N_{jk|l|m|n} = c_{lmn}N_{jk} + c_{lmn}\frac{1}{n+1}\{\dot\partial_j[(H_{ik}-H_{ki})y^i] - (H_{jk}-H_{kj})\} + d_{lmn}\,(1-n)g_{jk} - \frac{1}{n+1}\{\dot\partial_j[(H_{ik}-H_{ki})y^i] - (H_{jk}-H_{kj})\}_{|l|m|n},$$

Using (1.11) in (3.18), we get

(3.19) $$N_{jk|l|m|n} = c_{lmn}N_{jk} + d_{lmn}\,(1-n)g_{jk}$$

if and only if

(3.20) $$\{(1-n)\dot\partial_j\dot\partial_k H + H_{jk} + H_{kj}\}_{|l|m|n} = c_{lmn}\frac{1}{n+1}\{(1-n)\dot\partial_j\dot\partial_k H + H_{jk} + H_{kj}\}$$

Thus, we conclude

Corollary 3.1. *In* $GH^{hv}-(M)TRF_n$, *the Ricci tensor* N_{jk} *of the normal projective curvature tensor* N^i_{jkh} *is non – vanishing if and only if the tensor* $[(1-n)\dot\partial_j\dot\partial_k H + H_{jk} + H_{kj}]$ *is mixed trirecurrent.*

IV. CONCLUSION

We introduced a Finsler space which Berwald curvature tensor H^i_{jkh} satisfies the generalized trirecurrence property by using the first and second kind of covariant derivatives simultaneously in Cartan sense. Further, we obtained the necessary and sufficient condition of Berwald curvature tensor H^i_{jkh}, Ricci tensor H_{jk}, normal curvature tensor N^i_{jkh} and Ricci tensor N_{jk} in $GH^{hv}-(M)TRF_n$.

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