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Application N −Fractional Calculus for Ordinary and Partial Differential Equations of N^{th} Order

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Abstract: *There are many papers for application of fractional calculus and Shehu transform to fractional differential equations (see [6], [9], [11] and [12]).*

The main objective of this paper is to apply N-fractional calculus to ordinary and partial differential equations of n^{th} order after converting them to fractional differential equations and also solved partial differential equations of n^{th} order containing three and four variables. Also, we obtained special cases of Gauss's ordinary and partial differential equations..

Keywords*:* N-fractional calculus, the non-homogeneous n^th ordinary differential equation, the homogeneous n^{th} order ordinary differential equation, partial differential equations of order in two, three and four variables, fractional differential equation and Gauss's ordinary and partial differential equations

I. INTRODUCTION

The ordinary differential equations of n^{th} order play a crucial role in modeling many phenomena in science and engineering. Also the partial differential equations of n^{th} order are used to describe a wide range of physical phenomena. In this paper, we used fractional calculus defined by Nishimoto, K. of order (υ) for functions as the following (see [7]):

Definition: Let $D=\{D_-,\ D_+\}$, $C=\{\ C_-\ ,\ C_+\}$, where

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- ,

D_+ be a domain surrounded by C_+ .

(Here D contains the points over curves C).

Moreover, let $f=f(z)$ be a regular function in D ($z\in D$), $f_\upsilon = {}_C(f)_\nu(z) = \dfrac{\Gamma(\upsilon+1)}{2\pi i}\displaystyle\int_C \frac{f(\varsigma)}{(\varsigma-z)^{\upsilon+1}}\,d\varsigma$,

$(\upsilon \notin Z^-)$ (1.1)

$$(f)_{-m} = \lim_{\nu\to -m} (f)_\nu \quad (m\in Z^+),$$

where $\varsigma \neq z$, $z\in C$, $\upsilon \in R$, Γ :Gamma function, $-\pi \le \arg\,(\varsigma - z) \le \pi$ for C_-,

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$0 \leq \arg\ (\varsigma - z)\ \leq 2\pi$ for C_+, then $(f)_\upsilon$ is the fractional differintegration of arbitrary order υ (derivatives of order υ for $\upsilon>0$, and integrals of order $-\upsilon$ for $\upsilon<0$), with respect to z, of the function f, if $\left|(f)_\upsilon\right| < \infty$.

By using (1.1) we get the following Lemmas:

Lemma 1.: If k is constant, then

$$(k)_\alpha = k \cdot (1)_\alpha = 0\text{, if } \alpha \notin Z^- \cup \{0\} \quad (1.2)$$

Lemma 2.: Let $u = u(z)$, $y = y(z)$ be regular functions and if u_α, y_α exist, then

$$(au + by)_\alpha = a(u)_\alpha + b(\ y)_\alpha\text{, where } a, b \text{ are constants } (z, \alpha \in C). \quad (1.3)$$

Lemma 3.: Let $f = f(z)$ be regular function, if $(f_\mu)_\upsilon$ and $(f_\upsilon)_\mu$ exist, then

$$\left(f_\mu(z)\right)_\upsilon = \frac{\Gamma(\upsilon+1)}{2\pi i}\int_c \frac{f_\mu(\zeta)}{(\zeta - z)^{\upsilon+1}} d\zeta\ , \quad (z, \upsilon, \mu \in C)$$

$$\left(f_\mu(z)\right)_\upsilon = \left(f_\upsilon(z)\right)_\mu$$

$$\left(f_\mu(z)\right)_\upsilon = (f)_{\upsilon+\mu}. \ (1.4)$$

Lemma 4.: Let $u = u(z)$, $y = y(z)$ be regular functions and if are u_α, y_α exist, then

$$(u\ y)_\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - n + 1)\Gamma(n+1)} (y)_{\alpha-n}(u)_n, \quad (z, \alpha \in C). (1.5)$$

II. APPLICATIONS ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Fractional calculus can be used to solve (ODEs) of n^{th} order that contain derivatives of fractional order. these equations appear in many applications, such as:

1- Physics:

(i) Planetary motions: Newton's lows of motion and the low of universal gravitation lead second order differential equations that describe the motion of planets around the sun.

(ii) Projectile motion: Differential equations can be used to analyze the motion of projectiles in a gravitational field.

(iii) Oscillations: Differential equations can be used to describe the motion of oscillating systems, such as springs and pendulums.

2- Engineering:

(i) Electrical circuits: Differential equations can be used to analyze the behavior of electrical circuits containing resistors, inductors and capacitors.

(ii) Automatic control: Differential equations are used in the design of automatic control systems, such as temperature control systems and robots.

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3- Biology:

(i) Population growth: Differential equations can be used to describe population growth and changes in the numbers of living organisms over time.

(ii) Epidemics: Differential equations can be used to model the spread of epidemics and infectious diseases.

4- Economics:

(i) Economic growth: differential equations can be used to describe economic growth and changes in gross domestic product over time.

(ii) Financial markets: differential equations can be used to analyze the behavior of financial markets and changes in stock and bond prices.

The partial differential equations of n^{th} order are used in many applications, including:

1- Particle physics: Used to model the behavior of subatomic particles.

2- String theory: Used to describe superstrings.

3- General relativity: Used to describe specetime and its curvature.

III. PREVIOUS STUDIES TO ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

In this part, we review some previous studies in the same research direction as the following:

Nishimoto applied N-fractional calculus to ordinary and partial differential equations of second, third, fourth and fifth order (see [7] and [8]). For example, Nishimoto solved Gauss's ordinary differential equations (homogeneous and nonhomogeneous) as the following (see [7], Vol. 4, p. 106-109):

Theorem (3.1): If $f_{-\alpha}$ exists and $f_{-\alpha} \neq 0$, then the differential equation

$$\varphi_2 \cdot (z^2 - z) + \varphi_1 \cdot \{z \cdot (\alpha + \beta + 1) - \gamma\} + \varphi \cdot \alpha \cdot \beta = f, \ (z \neq 0, \ 1) \quad (3.1)$$

has a particular solution of the form

$$\varphi = w_\alpha = \left[\left(f_{-\alpha} \cdot z^{\gamma-\alpha-1} \cdot (z-1)^{\beta-\gamma} \right)_{-1} \cdot z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1} \right]_{\alpha-1}, \quad (3.2)$$

where $\varphi = \varphi(z)$ and α, β, γ are constants.

Theorem (3.2): the differential equation

$$\varphi_2 \cdot (z^2 - z) + \varphi_1 \cdot \{z \cdot (\alpha + \beta + 1) - \gamma\} + \varphi \cdot \alpha \cdot \beta = 0, \ (z \neq 0, \ 1) \quad (3.3)$$

has a particular solution of the form

$$\varphi = w_\alpha = \left[m \cdot z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1} \right]_{\alpha-1}, \quad (3.4)$$

where $\varphi = \varphi(z)$ and m, α, β, γ are constants.

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Theorem (3.3): If $f_{-\alpha} \neq 0$, then the differential equation (3.1) has the solution

$$\varphi = w_\alpha = \left[\left(f_{-\alpha} \cdot z^{\gamma-\alpha-1} \cdot (z-1)^{\beta-\gamma}\right)_{-1} \cdot z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1}\right]_{\alpha-1} + \left[m \cdot z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1}\right]_{\alpha-1}$$

(3.5)

for $\alpha \notin Z^- \cup \{0\}$ and m, α, β, γ are constants.

Also, Nishimoto solved Gauss's partial differential equation in only two variables as the following (see [7], Vol. 3, p. 129-131):

Theorem (3.4): Gauss's partial differential equation

$$\frac{\partial^2 u}{\partial z^2} \cdot (z^2 - z) + \frac{\partial u}{\partial z} \cdot \{z \cdot (\alpha + \beta + 1) - \gamma\} + u = A_1 \frac{\partial^2 u}{\partial t^2} + B_1 \frac{\partial u}{\partial t}, \quad (z \neq 0,\ 1) \quad (3.6)$$

has a particular solution of the form (i) $u(z,t) = \left[m \cdot z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1} \cdot e^{\frac{-B_1 \pm \sqrt{B_1^2 + 4A_1(1-\alpha\cdot\beta)}}{2A_1} \cdot t}\right]_{\alpha-1}$, for

$A_1 \cdot B_1 \neq 0.$ (3.7)

(ii) $u(z,t) = \left[m \cdot z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1} \cdot e^{\pm\sqrt{\frac{(1-\alpha\cdot\beta)}{A_1}} \cdot t}\right]_{\alpha-1}$, for $A_1 \neq 0$ and $B_1 = 0$.

(iii) $u(z,t) = \left[m \cdot z^{\alpha-\gamma} \cdot (z-1)^{\gamma-\beta-1} \cdot e^{\frac{(1-\alpha\cdot\beta)}{B_1} \cdot t}\right]_{\alpha-1}$, for $A_1 = 0$ and $B_1 \neq 0$.

Where $\alpha = \alpha(z)$, $\alpha \notin Z^- \cup \{0\}$ and $m, \alpha, \beta, \gamma, A_1, B_1$ are constants. Here, Nishimoto has stopped. In [1] - [3], Bassim improved Nishimoto's method to solving partial differential equations of second, third, fourth, and fifth order in two variables to solving partial differential equations of three and four variables. This method enables the researcher to get solutions of partial differential equations of five, six, … , etc variables. Also, Bassim solved ordinary differential equations of n^{th} order by using N-fractional calculus (see [4] and [5]). Shih-Tong Tu and Ding-Kuo solved ordinary and partial differential equations of n^{th} order in only two variables by using N-fractional calculus (see [10]).

IV. THE RESULTS BY USING N-FRACTIONAL CALCULUS

In this section, we present the theorems we obtained as follows:

Theorem (4.1): If $f_{-\alpha}$ exists and $f_{-\alpha} \neq 0$, then the non-homogeneous n^{th} order linear ordinary differential equation

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$$\sum_{k=0}^{n}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\varphi_{n-k}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\varphi_{n-k-1}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}=f,\ (\ z\neq 0 \text{ and } z\neq 1\ )(4.1)$$

has a particular solution of the form

$$\varphi=w_\alpha=\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1},\quad(4.2)$$

where $\varphi=\varphi(z)$, $f=f(z)$, $z\in C$ and b_{n-1} and b_n are constants.

Theorem (4.2): The homogeneous n^{th} order linear ordinary differential equation

$$\sum_{k=0}^{n}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\varphi_{n-k}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\varphi_{n-k-1}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}=0,\ (\ z\neq 0 \text{ and } z\neq 1\ )\ (4.3)$$

has a solution of the form

$$\varphi=w_\alpha=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1},\quad(4.4)$$

for $\alpha\notin Z^-\cup\{0\}$, $\varphi=\varphi(z)$, $z\in C$ and m, b_{n-1} and b_n are constants.

Theorem (4.3): If $f_{-\alpha}$ exists and $f_{-\alpha}\neq 0$, then the differential equation (4.1) has the solution

$$\varphi=w_\alpha=\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1}$$

$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1},\quad \text{for } \alpha\notin Z^-\cup\{0\}.\quad(4.5)$$

Also, we solved partial differential equations of n^{th} order by using N-fractional calculus in two, three and four variables and we could continue to more than four variables but we stopped here.

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Theorem (4.4): The partial differential equation of n^{th} order

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\frac{\partial^{n-k}u}{\partial z^{n-k}}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\frac{\partial^{n-k-1}u}{\partial z^{n-k-1}}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\cdot\frac{\partial u}{\partial z}\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}$$

$$+u=A_1\frac{\partial^2u}{\partial t^2}+B_1\frac{\partial u}{\partial t},\ (\ z\neq 0 \text{ and } z\neq 1\ )\quad(4.6)$$

which has the following solutions

$$(i)\quad u(z,t)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t}\right]_{\alpha-n+1},\quad \text{for } A_1\cdot B_1\neq 0.\quad(4.7)$$

$$(ii)\quad u(z,t)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\pm\sqrt{\frac{\sigma}{A_1}}\cdot t}\right]_{\alpha-n+1},\quad \text{for } A_1\neq 0 \text{ and } B_1=0.$$

$$(iii)\quad u(z,t)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{\sigma}{B_1}\cdot t}\right]_{\alpha-n+1},\quad \text{for } A_1=0 \text{ and } B_1\neq 0.$$

Where $\alpha=\alpha(z)$, $\alpha\notin Z^-\cup\{0\}$, $\sigma=1-\dfrac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)}$. and m, b_{n-1} and b_n are constants.

Theorem (4.5): The partial differential equation of n^{th} order

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\frac{\partial^{n-k}\Psi}{\partial z^{n-k}}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\frac{\partial^{n-k-1}\Psi}{\partial z^{n-k-1}}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\cdot\frac{\partial\Psi}{\partial z}\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}$$

$$+\Psi-A_1\frac{\partial^2\Psi}{\partial t^2}-B_1\frac{\partial\Psi}{\partial t}=A_2\frac{\partial^2\Psi}{\partial x^2}+B_2\frac{\partial\Psi}{\partial x},\ (\ z\neq 0 \text{ and } z\neq 1\ )(4.8)$$

has the solutions of the form

$$(i)\Psi(z,t,x)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1},\ \text{for } A_1\cdot B_1\neq 0.\quad(4.9)$$

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$$(ii)\Psi(z,t,x)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\pm\sqrt{\frac{\sigma}{A_1}}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-n+1}, \text{ for } A_1\neq 0 \text{ and } B_1=0.$$

$$(iii)\Psi(z,t,x)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{\sigma}{B_1}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-n+1}, \text{ for } A_1=0 \text{ and } B_1\neq 0.$$

Where $\alpha=\alpha(z)$, $\alpha\notin Z^-\cup\{0\}$, $\sigma=1-\dfrac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)}$. and m, b_{n-1} and b_n are constants.

Theorem (4.6): The partial differential equation of n^{th} order

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\frac{\partial^{n-k}\varpi}{\partial z^{n-k}}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\frac{\partial^{n-k-1}\varpi}{\partial z^{n-k-1}}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\cdot\frac{\partial\varpi}{\partial z}\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}+\varpi-A_1\frac{\partial^2\varpi}{\partial t^2}$$

$$-B_1\frac{\partial\varpi}{\partial t}-A_2\frac{\partial^2\varpi}{\partial x^2}-B_2\frac{\partial\varpi}{\partial x}=A_3\frac{\partial^2\varpi}{\partial y^2}+B_3\frac{\partial\varpi}{\partial y},\ (z\neq 0 \text{ and } z\neq 1)\quad(4.10)$$

has the solutions of the form

$$(i)\varpi(z,t,x,y)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x-\frac{B_3}{A_3}\cdot y}\right]_{\alpha-n+1}, \text{ for } A_1\cdot B_1\neq 0.$$

(4.11)

$$(ii)\varpi(z,t,x,y)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\pm\sqrt{\frac{\sigma}{A_1}}\cdot t-\frac{B_2}{A_2}x-\frac{B_3}{A_3}y}\right]_{\alpha-n+1}, \text{for } A_1\neq 0 \text{ and } B_1=0.$$

$$(iii)\varpi(z,t,x,y)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{\sigma}{B_1}\cdot t-\frac{B_2}{A_2}x-\frac{B_3}{A_3}y}\right]_{\alpha-n+1}, \text{ for } A_1=0 \text{ and } B_1\neq 0.$$

Where $\alpha=\alpha(z)$, $\alpha\notin Z^-\cup\{0\}$, $\sigma=1-\dfrac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)}$. and m, b_{n-1} and b_n are constants.

Also, Nishimoto solved Gauss's partial differential equation in two variables only. But we solved it in three and four variables as the following:

Theorem (4.7): Gauss's partial differential equation in three variables

$$\frac{\partial^2\Psi}{\partial z^2}\cdot(z^2-z)+\frac{\partial\Psi}{\partial z}\cdot\{z\cdot(\alpha+\beta+1)-\gamma\}+\Psi-A_1\frac{\partial^2\Psi}{\partial t^2}-B_1\frac{\partial\Psi}{\partial t}=A_2\frac{\partial^2\Psi}{\partial x^2}+B_2\frac{\partial\Psi}{\partial x},$$

$(z\neq 0,\ 1)$(4.12)

has the particular solutions of the form

$$(i)\quad \Psi(z,t,x)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1(1-\alpha\cdot\beta)}}{2A_1}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-1},\ \text{for } A_1\cdot B_1\neq 0.\ (4.13)$$

$$(ii)\quad \Psi(z,t,x)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\pm\sqrt{\frac{(1-\alpha\cdot\beta)}{A_1}}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-1},\ \text{for } A_1\neq 0 \text{ and } B_1=0.$$

$$(iii)\quad \Psi(z,t,x)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\frac{(1-\alpha\cdot\beta)}{B_1}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-1},\ \text{for } A_1=0 \text{ and } B_1\neq 0.$$

Where $\alpha=\alpha(z)$, $\alpha\notin Z^-\cup\{0\}$ and $m,\ \alpha,\beta,\gamma$ are constants.

Theorem (4.8): Gauss's partial differential equation in four variables

$$\frac{\partial^2\varpi}{\partial z^2}\cdot(z^2-z)+\frac{\partial\varpi}{\partial z}\cdot\{z\cdot(\alpha+\beta+1)-\gamma\}+\varpi-A_1\frac{\partial^2\varpi}{\partial t^2}-B_1\frac{\partial\varpi}{\partial t}-A_2\frac{\partial^2\varpi}{\partial x^2}-B_2\frac{\partial\varpi}{\partial x}=$$

$$A_3\frac{\partial^2\varpi}{\partial y^2}+B_3\frac{\partial\varpi}{\partial y},(z\neq 0,\ 1)\ (4.14)$$

has the particular solutions of the form

$$(i)\varpi(z,t,x,y)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1(1-\alpha\cdot\beta)}}{2A_1}\cdot t-\frac{B_2}{A_2}x-\frac{B_3}{A_3}y}\right]_{\alpha-1},\ \text{for } A_1\cdot B_1\neq 0.(4.15)$$

$$(ii)\varpi(z,t,x,y)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\pm\sqrt{\frac{(1-\alpha\cdot\beta)}{A_1}}\cdot t-\frac{B_2}{A_2}x-\frac{B_3}{A_3}y}\right]_{\alpha-1},\ \text{for } A_1\neq 0 \text{ and } B_1=0.$$

$$(iii)\varpi(z,t,x,y)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\frac{(1-\alpha\cdot\beta)}{B_1}\cdot t-\frac{B_2}{A_2}x-\frac{B_3}{A_3}y}\right]_{\alpha-1},\ \text{for } A_1=0 \text{ and } B_1\neq 0.$$

Where $\alpha=\alpha(z)$, $\alpha\notin Z^-\cup\{0\}$and $m,\ \alpha,\beta,\gamma$ are constants.

V. PROOF OF THE RESULTS

In this part, we prove the theorems in section(4). But we can not solve differential equations of the natural order by using N-fractional calculus, so we must convert the differential equations from the natural order to the fractional order as the following:

Proof theorem (4.1):

Put $\varphi_n = w_{\alpha+n}$, where $\alpha \in]0,1[$ and $w = w(z)$. (5.1)

Substituting (5.1) into (4.1); we get

$$\sum_{k=0}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \Big\{ w_{n+\alpha-k} \cdot \left(z\cdot(z-1)^{n-1}\right)_k + w_{n+\alpha-k-1} \cdot \Big[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k$$

$$+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k \Big]\Big\} = f, \ (\ z \neq 0 \text{ and } z \neq 1 \ )(5.2)$$

The equation (5.2) is fractional differential equation of order $n+\alpha$.

Now, by using (1.3) - (1.5), then (5.2) becomes

$$w_n z\cdot(z-1)^{n-1} + (-b_{n-1}+n-2)w_{n-1}(z-1)^{n-1} + (b_{n-1}+b_n+n-2)w_{n-1}z\cdot(z-1)^{n-2} = f_{-\alpha},$$

(5.3)

Multiply (5.3) by $z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}$ and by using (1.4); we obtain

$$w_{n-1} = \left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3} \ (z-1)^{b_{n-1}+b_n-1} \right)_{-1} \cdot z^{b_{n-1}-n+2}(z-1)^{-b_{n-1}-b_n-n+2},$$

this is

$$\varphi = w_\alpha = \left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1} \right)_{-1} \cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-n+1},$$

as solution of the non-homogeneous n^{th} order linear ordinary differential equation (4.1). Here completes the proof of the theorem (4.1).

Inversely, we have

$$\varphi = w_\alpha = \left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1} \right)_{-1} \cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-n+1}. \quad (5.4)$$

Substituting (5.4) into the left hand side of (4.1); we get

$$L.\ H.\ S.\ (4.1) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)}\cdot$$

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$$\left\{ \left[\left(f_{-\alpha} \cdot z^{-b_{n-1}+n-3} \cdot (z-1)^{b_{n-1}+b_n-1} \right)_{-1} \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-k+1} . \left(z(z-1)^{n-1} \right)_k \right.$$

$$+ (-b_{n-1}+n-2) \left[\left(f_{-\alpha} \cdot z^{-b_{n-1}+n-3} \cdot (z-1)^{b_{n-1}+b_n-1} \right)_{-1} \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-k} . \left((z-1)^{n-1} \right)_k$$

$$\left. + (b_{n-1}+b_n+n-2) \left[\left(f_{-\alpha} \cdot z^{-b_{n-1}+n-3} \cdot (z-1)^{b_{n-1}+b_n-1} \right)_{-1} \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-k} . \left(z(z-1)^{n-2} \right)_k \right\}.$$

By using (1.3) - (1.5); we obtain

L. H. S. (4.1) = f = R. H. S. (4.1).

Proof theorem (4.2):

Here, we prove to the homogeneous n^{th} order linear ordinary differential equation (4.3) its solution is (4.4) as the following:

Put $\varphi_n = w_{\alpha+n}$ where $\alpha \in]0,1[$ in (4.3), we get:

$$\sum_{k=0}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \left\{ w_{n+\alpha-k} \cdot \left(z\cdot(z-1)^{n-1} \right)_k + w_{n+\alpha-k-1} \cdot \left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1} \right)_k \right.\right.$$

$$\left.\left. + (b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2} \right)_k \right] \right\} = 0, \ (z \neq 0 \text{ and } z \neq 1) \ (5.5)$$

Now, by using (1.2) - (1.5), then (5.5) becomes

$$w_n z(z-1)^{n-1} + (-b_{n-1}+n-2) w_{n-1}(z-1)^{n-1} + (b_{n-1}+b_n+n-2) w_{n-1} z(z-1)^{n-2} = 0, \ (5.6)$$

for $\alpha \notin Z^- \cup \{0\}$.

Multiply (5.6) by $z^{-b_{n-1}+n-3} \cdot (z-1)^{b_{n-1}+b_n-1}$ and by using (1.4); we obtain

$$w_{n-1} = m \cdot z^{b_{n-1}-n+2} (z-1)^{-b_{n-1}-b_n-n+2},$$

this is

$$\varphi = w_\alpha = \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-n+1},$$

as solution of the homogeneous n^{th} order linear ordinary differential equation (4.3).

The intended outcome is achieved.

Inversely, we have

$$\varphi = w_\alpha = \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-n+1}. \quad (5.7)$$

Substituting (5.7) into the left hand side of (4.3); we get

$$L.\ H.\ S.\ (4.3) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)} \cdot$$

$$\Bigg\{ \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-k+1} . \left(z(z-1)^{n-1}\right)_k$$

$$+(-b_{n-1}+n-2)\left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-k} . \left((z-1)^{n-1}\right)_k$$

$$+(b_{n-1}+b_n+n-2)\left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \right]_{\alpha-k} . \left(z(z-1)^{n-2}\right)_k \Bigg\}$$

By using (1.3) - (1.5); we obtain

L. H. S. (4.3) $= 0 =$ R. H. S. (4.3).

Proof theorem (4.3):

Put $\varphi_n = w_{\alpha+n}$, where $\alpha \in]0,1[$. (5.8)

Substituting (5.8) into (4.1); we get

$$\sum_{k=0}^{n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \Bigg\{ w_{n+\alpha-k} \cdot \left(z\cdot(z-1)^{n-1}\right)_k + w_{n+\alpha-k-1} \cdot \Big[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k$$

$$+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k \Big]\Bigg\} = f, \ (\ z \neq 0 \text{ and } z \neq 1\ )(5.9)$$

Now, by using (1.3) - (1.5), then (5.9) becomes

$$w_n z\cdot(z-1)^{n-1} + (-b_{n-1}+n-2)w_{n-1}(z-1)^{n-1} + (b_{n-1}+b_n+n-2)w_{n-1}z\cdot(z-1)^{n-2} = f_{-\alpha},$$

(5.10)

Multiply (5.10) by $z^{-b_{n-1}+n-3} \cdot (z-1)^{b_{n-1}+b_n-1}$ and hence using(1.2); we get

$$\left(w_{n-1} \cdot z^{-b_{n-1}+n-2} \ \ (z-1)^{b_{n-1}+b_n+n-2} \right)_1 = f_{-\alpha} \cdot z^{-b_{n-1}+n-3}(z-1)^{b_{n-1}+b_n-1} + (m)_1,$$

for $\alpha \notin Z^- \cup \{0\}$.

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This implies that

$$w_{n-1}=\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}(z-1)^{b_{n-1}+b_n-1}\right)_{-1}z^{b_{n-1}-n+2}(z-1)^{-b_{n-1}-b_n-n+2}+m\cdot z^{b_{n-1}-n+2}(z-1)^{-b_{n-1}-b_n-n+2}$$, this is

$$\varphi=w_\alpha=\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1}$$

$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1},\quad \text{for } \alpha\notin Z^-\cup\{0\},$$

as solution of the non-homogeneous n^{th} order linear ordinary differential equation (4.1). Now, we arrive to the proof of the theorem (4.3).

Inversely, we have

$$\varphi=w_\alpha=\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1}$$

$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1},$$

or,

$$\varphi=w_\alpha=\left[\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}+m\right]\cdot z^{b_{n-1}-n+2}(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-n+1}. \tag{5.11}$$

Substituting (5.11) into the left hand side of (4.1); we get

$$L.\ H.\ S.\ (4.1)=\sum_{k=0}^{n}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)}\cdot$$

$$\left\{\left[\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}+m\right]z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-k+1}.\left(z(z-1)^{n-1}\right)_k+\right.$$

$$+(-b_{n-1}+n-2)\left[\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}+m\right]z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-k}.\left((z-1)^{n-1}\right)_k$$

$$\left.+(b_{n-1}+b_n+n-2)\left[\left[\left(f_{-\alpha}\cdot z^{-b_{n-1}+n-3}\cdot(z-1)^{b_{n-1}+b_n-1}\right)_{-1}+m\right]z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\right]_{\alpha-k}.\left(z(z-1)^{n-2}\right)_k\right\}.$$

By using (1.3) - (1.5); we obtain

L. H. S. (4.1) $=f=$ R. H. S. (4.1).

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Proof theorem (4.4):

Now, we will prove theorem (4.4) as the following:

Put $u(z,t)=\varphi(z)\cdot e^{\mu\cdot t},\ (\mu\neq 0)$ (5.12)

then,

$$\frac{\partial u}{\partial z}=\varphi_1(z)\cdot e^{\mu\cdot t},\ \frac{\partial^2 u}{\partial z^2}=\varphi_2(z)\cdot e^{\mu\cdot t},\ldots,\ \frac{\partial^{n-k-1}u}{\partial z^{n-k-1}}=\varphi_{n-k-1}(z)\cdot e^{\mu\cdot t},\ \frac{\partial^{n-k}u}{\partial z^{n-k}}=\varphi_{n-k}(z)\cdot e^{\mu\cdot t}.$$

(5.13)

and

$$\frac{\partial u}{\partial t}=\varphi(z)\cdot\mu\cdot e^{\mu\cdot t},\ \frac{\partial^2 u}{\partial t^2}=\varphi(z)\cdot\mu^2\cdot e^{\mu\cdot t},\ (5.14)$$

Substituting (5.12) - (5.14) in (4.6); we get

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\Big\{\varphi_{n-k}\cdot e^{\mu\cdot t}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\varphi_{n-k-1}\cdot e^{\mu\cdot t}\cdot\Big[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k$$

$$+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\Big]\Big\}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\cdot\varphi_1\cdot e^{\mu\cdot t}\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}$$

$$+\varphi\cdot e^{\mu\cdot t}=A_1\cdot\varphi\cdot\mu^2\cdot e^{\mu\cdot t}+B_1\cdot\varphi\cdot\mu\cdot e^{\mu\cdot t},$$

this is,

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\Big\{\varphi_{n-k}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\varphi_{n-k-1}\cdot\Big[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k$$

$$+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\Big]\Big\}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\cdot\varphi_1\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}$$

$$+\varphi\cdot(-A_1\cdot\mu^2-B_1\cdot\mu+1)=0,\ (5.15)$$

comparing the equation (5.15) with (4.3); we get

$$-A_1\cdot\mu^2-B_1\cdot\mu+1=\frac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)},$$

or,

$$A_1\cdot\mu^2+B_1\cdot\mu=1-\frac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)},\ (5.16)$$

and

$A_1\cdot\mu^2+B_1\cdot\mu-\sigma=0,$ where $\sigma=1-\dfrac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)}.$

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which its solution for μ is $\mu = \dfrac{-B_1 \pm \sqrt{B_1^{\,2} + 4A_1\sigma}}{2A_1}$.

This implies that the solutions of the equation (4.6) are

$$(i) \quad u(z,t) = \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \cdot e^{\frac{-B_1 \pm \sqrt{B_1^{\,2}+4A_1\sigma}}{2A_1} \cdot t} \right]_{\alpha-n+1}, \quad \text{for } A_1 \cdot B_1 \neq 0.$$

$$(ii) \quad u(z,t) = \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \cdot e^{\pm\sqrt{\frac{\sigma}{A_1}} \cdot t} \right]_{\alpha-n+1}, \quad \text{for } A_1 \neq 0 \text{ and } B_1 = 0.$$

$$(iii) \quad u(z,t) = \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \cdot e^{\frac{\sigma}{B_1} \cdot t} \right]_{\alpha-n+1}, \quad \text{for } A_1 = 0 \text{ and } B_1 \neq 0.$$

Where $\alpha = \alpha(z)$, $\alpha \notin Z^- \cup \{0\}$, $\sigma = 1 - \dfrac{(\alpha + b_n + n - 3) \cdot \Gamma(\alpha+1)}{\Gamma(\alpha - n + 2)}$. and m, b_{n-1} and b_n are constants. Thus, we achieve the desired outcome.

Inversely, we have

$$u(z,t) = \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \cdot e^{\frac{-B_1 \pm \sqrt{B_1^{\,2}+4A_1\sigma}}{2A_1} \cdot t} \right]_{\alpha-n+1}. \tag{5.17}$$

Substituting (5.17) into the left hand side of (4.6); we get

L. H. S. (4.6)=

$$\sum_{k=0}^{n-2} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)} \left\{ \left[\left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \cdot e^{\frac{-B_1 \pm \sqrt{B_1^{\,2}+4A_1\sigma}}{2\cdot A_1} \cdot t} \right]_{\alpha-k+1} \cdot \left(z \cdot (z-1)^{n-1} \right)_k + \right. \right.$$

$$+ \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \cdot e^{\frac{-B_1 \pm \sqrt{B_1^{\,2}+4\cdot A_1\sigma}}{2\cdot A_1} \cdot t} \right]_{\alpha-k} \cdot \left[(-b_{n-1} + n - 2) \cdot \left((z-1)^{n-1} \right)_k \right.$$

$$\left. \left. + (b_{n-1} + b_n + n - 2) \cdot \left(z \cdot (z-1)^{n-2} \right)_k \right] \right\} +$$

$$+ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n} \left[m \cdot z^{b_{n-1}-n+2} \cdot (z-1)^{-b_{n-1}-b_n-n+2} \cdot e^{\frac{-B_1 \pm \sqrt{B_1^{\,2}+4A_1\sigma}}{2A_1} \cdot t} \right]_{\alpha-n+2} . \left(z \cdot (z-1)^{n-1} \right)_{n-1}$$

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$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t}\right]_{\alpha-n+1}. \quad (5.18)$$

by using (1.3) - (1.5), then (5.18) becomes

$$\text{L. H. S. (4.6)}=\left(\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t}\right]_1\cdot z\cdot(z-1)^{n-1}+\right.$$

$$(-b_{n-1}+n-2)\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n+1}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t}\right]+$$

$$\left.(b_{n-1}+b_n+n-2)\cdot\left[m\cdot z^{b_{n-1}-n+3}\cdot(z-1)^{-b_{n-1}-b_n}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t}\right]\right)_\alpha+$$

$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t}\right]_{\alpha-n+1}$$

$$-\frac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)}\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t}\right]_{\alpha-n+1}. \quad (5.19)$$

by using (1.2), (5.16) and (5.17) then (5.19) becomes

L. H. S. of (4.6) = R. H. S. of (4.6)

Proof theorem (4.5):

We can prove theorem (4.5) as the following:

Put $\Psi(z,t,x)=u(z,t)\cdot e^{\lambda\cdot x},\ (\lambda\neq 0)$ (5.20)

then

$$\frac{\partial\Psi}{\partial z}=\frac{\partial u}{\partial z}\cdot e^{\lambda\cdot x},\ \frac{\partial^2\Psi}{\partial z^2}=\frac{\partial^2 u}{\partial z^2}\cdot e^{\lambda\cdot x},\ldots,\ \frac{\partial^{n-k-1}\Psi}{\partial z^{n-k-1}}=\frac{\partial^{n-k-1}u}{\partial z^{n-k-1}}\cdot e^{\lambda\cdot x},\ \frac{\partial^{n-k}\Psi}{\partial z^{n-k}}=\frac{\partial^{n-k}u}{\partial z^{n-k}}\cdot e^{\lambda\cdot x}.$$

(5.21)

And

$$\frac{\partial\Psi}{\partial t}=\frac{\partial u}{\partial t}\cdot e^{\lambda\cdot x},\ \frac{\partial^2\Psi}{\partial t^2}=\frac{\partial^2 u}{\partial t^2}\cdot e^{\lambda\cdot x}, \quad (5.22)$$

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$$\frac{\partial \Psi}{\partial x} = u\cdot\lambda\cdot e^{\lambda\cdot x},\quad \frac{\partial^2 \Psi}{\partial x^2} = u\cdot\lambda^2\cdot e^{\lambda\cdot x}, \quad (5.23)$$

Substituting (5.20) - (5.23) in (4.8); we get

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\frac{\partial^{n-k}u}{\partial z^{n-k}}\cdot e^{\lambda\cdot x}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\frac{\partial^{n-k-1}u}{\partial z^{n-k-1}}\cdot e^{\lambda\cdot x}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\cdot\frac{\partial u}{\partial z}\cdot e^{\lambda\cdot x}\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}$$

$$+u\cdot e^{\lambda\cdot x}-A_1\frac{\partial^2 u}{\partial t^2}\cdot e^{\lambda\cdot x}-B_1\frac{\partial u}{\partial t}\cdot e^{\lambda\cdot x}=A_2\cdot u\cdot\lambda^2\cdot e^{\lambda\cdot x}+B_2\cdot u\cdot\lambda\cdot e^{\lambda\cdot x},$$

this is,

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\frac{\partial^{n-k}u}{\partial z^{n-k}}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\frac{\partial^{n-k-1}u}{\partial z^{n-k-1}}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\cdot\frac{\partial u}{\partial z}\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}$$

$$+u\cdot(1-A_2\cdot\lambda^2-B_2\cdot\lambda)=A_1\frac{\partial^2 u}{\partial t^2}+B_1\frac{\partial u}{\partial t}, (5.24)$$

comparing the equation (5.24) with (4.6); we get

$$A_2\cdot\lambda^2+B_2\cdot\lambda=0, \quad (5.25)$$

or,

$$\lambda=\frac{-B_2}{A_2}.$$

This implies that the solutions of the equation (4.8) are

$$(i)\quad \Psi(z,t,x)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1},\quad \text{for } A_1\cdot B_1\neq 0.$$

$$(ii)\quad \Psi(z,t,x)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\pm\sqrt{\frac{\sigma}{A_1}}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1},\quad \text{for } A_1\neq 0 \text{ and } B_1=0.$$

$$(iii)\quad \Psi(z,t,x)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{\sigma}{B_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1},$$

for $A_1=0$ and $B_1\neq 0$ and $\sigma=1-\dfrac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)}.$

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We eventually arrive at the required solutions of the equation (4.8).

Inversely, we have

$$\Psi(z,t,x)=\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}. \tag{5.26}$$

Substituting (5.26) into the left hand side of (4.8); we get

L. H. S. (4.8)=

$$\sum_{k=0}^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\cdot\Gamma(k+1)}\left\{\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2\cdot A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-k+1}\cdot\left(z\cdot(z-1)^{n-1}\right)_k+\right.$$

$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4\cdot A_1\sigma}}{2\cdot A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-k}\cdot\left[(-b_{n-1}+n-2)\cdot\left((z-1)^{n-1}\right)_k\right.$$

$$\left.\left.+(b_{n-1}+b_n+n-2)\cdot\left(z\cdot(z-1)^{n-2}\right)_k\right]\right\}+$$

$$+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)\cdot\Gamma n}\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+2}\cdot\left(z\cdot(z-1)^{n-1}\right)_{n-1}$$

$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}$$

$$-A\cdot\mu^2\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}$$

$$-B\cdot\mu\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}. \tag{5.27}$$

By using (1.3) - (1.5), then (5.27) becomes

$$\text{L. H. S. (4.8)}=\left(\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_1\cdot z\cdot(z-1)^{n-1}+\right.$$

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$$(-b_{n-1}+n-2)\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n+1}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]+$$

$$\left.(b_{n-1}+b_n+n-2)\cdot\left[m\cdot z^{b_{n-1}-n+3}\cdot(z-1)^{-b_{n-1}-b_n}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]\right)_{\alpha}+$$

$$+\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}$$

$$-\frac{(\alpha+b_n+n-3)\cdot\Gamma(\alpha+1)}{\Gamma(\alpha-n+2)}\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}$$

$$-A\cdot\mu^2\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}$$

$$-B\cdot\mu\cdot\left[m\cdot z^{b_{n-1}-n+2}\cdot(z-1)^{-b_{n-1}-b_n-n+2}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1\sigma}}{2A_1}\cdot t-\frac{B_2}{A_2}\cdot x}\right]_{\alpha-n+1}. \quad (5.28)$$

By using (1.2), (5.23) and (5.25), then (5.28) becomes

L. H. S. of (4.8) = R. H. S. of (4.8)

Proof theorem (4.6):

We can prove theorem (4.6) by using same steps in proof theorem (4.5).

Proof theorem (4.7):

Put $\Psi(z,t,x)=u(z,t)\cdot e^{\lambda\cdot x},\ (\lambda\neq 0)$ (5.29)

then

$$\frac{\partial\Psi}{\partial z}=\frac{\partial u}{\partial z}\cdot e^{\lambda\cdot x},\quad \frac{\partial^2\Psi}{\partial z^2}=\frac{\partial^2 u}{\partial z^2}\cdot e^{\lambda\cdot x}, \quad (5.30)$$

and

$$\frac{\partial\Psi}{\partial t}=\frac{\partial u}{\partial t}\cdot e^{\lambda\cdot x},\quad \frac{\partial^2\Psi}{\partial t^2}=\frac{\partial^2 u}{\partial t^2}\cdot e^{\lambda\cdot x}, \quad (5.31)$$

$$\frac{\partial\Psi}{\partial x}=u\cdot\lambda\cdot e^{\lambda\cdot x},\quad \frac{\partial^2\Psi}{\partial x^2}=u\cdot\lambda^2\cdot e^{\lambda\cdot x}, \quad (5.32)$$

Substituting (5.29) - (5.32) in (4.12); we get

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$$\frac{\partial^2 u}{\partial z^2}\cdot(z^2-z)+\frac{\partial u}{\partial z}\cdot\{z\cdot(\alpha+\beta+1)-\gamma\}+u-A_1\frac{\partial^2 u}{\partial t^2}-B_1\frac{\partial u}{\partial t}=A_2\cdot u\cdot\lambda^2+B_2\cdot u\cdot\lambda,$$

$$\frac{\partial^2 u}{\partial z^2}\cdot(z^2-z)+\frac{\partial u}{\partial z}\cdot\{z\cdot(\alpha+\beta+1)-\gamma\}+u\cdot(1-A_2\cdot\lambda^2-B_2\cdot\lambda)=A_1\frac{\partial^2 u}{\partial t^2}-B_1\frac{\partial u}{\partial t}, \quad (5.33)$$

comparing the equation (5.33) with (3.6); we get

$$A_2\cdot\lambda^2+B_2\cdot\lambda=0 \quad (5.34)$$

or,

$$\lambda=\frac{-B_2}{A_2}.$$

This implies that the solutions of the equation (4.12) are

$$(i) \quad \Psi(z,t,x)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1(1-\alpha\cdot\beta)}}{2A_1}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-1}, \quad \text{for } A_1\cdot B_1\neq 0.$$

$$(ii) \quad \Psi(z,t,x)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\pm\sqrt{\frac{(1-\alpha\cdot\beta)}{A_1}}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-1}, \quad \text{for } A_1\neq 0 \text{ and } B_1=0.$$

$$(iii) \quad \Psi(z,t,x)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\frac{(1-\alpha\cdot\beta)}{B_1}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-1}, \quad \text{for } A_1=0 \text{ and } B_1\neq 0.$$

Where $\alpha=\alpha(z)$, $\alpha\notin Z^-\cup\{0\}$ and m,α,β,γ are constants.

Inversely, we have

$$\Psi(z,t,x)=\left[m\cdot z^{\alpha-\gamma}\cdot(z-1)^{\gamma-\beta-1}\cdot e^{\frac{-B_1\pm\sqrt{B_1^2+4A_1(1-\alpha\cdot\beta)}}{2A_1}\cdot t-\frac{B_2}{A_2}x}\right]_{\alpha-1}=u(z,t)\cdot e^{\frac{-B_2}{A_2}\cdot x}. \quad (5.35)$$

Substituting (5.35) into the left hand side of (4.12); we get

$$\text{L. H. S. (4.12)}=\left\{\frac{\partial^2 u}{\partial z^2}\cdot(z^2-z)+\frac{\partial u}{\partial z}\cdot\{z\cdot(\alpha+\beta+1)-\gamma\}+u-A_1\frac{\partial^2 u}{\partial t^2}-B_1\frac{\partial u}{\partial t}\right.$$

$$\left.-A_2\frac{\partial^2 u}{\partial x^2}-B_2\frac{\partial u}{\partial x}\right\}\cdot e^{\frac{-B_2}{A_2}\cdot x},$$

by using (3.6), (5.32) and (5.34), we obtain

L. H. S. of (4.12) = R. H. S. of (4.12)

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Proof theorem (4.8):

We can prove theorem (4.8) by using same steps in proof theorem (4.7).

VI. CONCLUSIONS

(i) Put $n=2,\ b_1=\alpha-\gamma$ and $b_2=\beta-\alpha+1$ in the theorems (4.1) - (4.3) respectively, we get Gauss's ordinary differential equations (homogeneous and nonhomogeneous) with their solutions in theorems (3.1) - (3.3) respectively.

(ii) Put $n=2,\ b_1=\alpha-\gamma$ and $b_2=\beta-\alpha+1$ in the theorem (4.4), we get Gauss's partial differential equation in two variables with its solutions in theorem (3.4).

(iii) Put $n=2,\ b_1=\alpha-\gamma$ and $b_2=\beta-\alpha+1$ in the theorems (4.5) and (4.6), we get Gauss's partial differential equations in three and four variables with their solutions in theorems (4.7) and (4.8).

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