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Important of Combinatorial Properties of Metric Spaces using Fixed Point Theory

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Abstract: *In this paper, we investigate some approximate fixed point results in extended b-metric spaces using several contraction mappings such as the Mohseni-Saheli contraction, the B-contraction, and their consequences. Additionally, we prove some ϵ-fixed point results by using rational type contraction mappings, Also, a few examples are included to illustrate the results. Finally, we discuss some applications that support our main results in the field of applied mathematics.*

Keywords*:* fixed point, b-metric space, approximate fixed point, rational contraction

I. INTRODUCTION

Using fixed point theory (f.p.t), researchers from many fields have contributed to the progress of science and technology. Large scale problems requiring f.p.t are highly esteemed for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing f.p.t approaches and have provided various useful techniques for discovering f.p's in complex issues. These are currently crucial in many mathematics related areas and its applications, including economics, astronomy, dynamical systems, decision theory, and parameter estimation. The father of f.p.t, mathematician Brouwer [1908], proposed f.p theorems (f.p.t′ s) for continuous mappings on finite dimensional space. Due to the complexity of finding an exact f.p, one can show the results of an approximate f.p (a.f.p). Because an exact f.p has overly strict limitations.

This is the main reason to find an a.f.p (ϵ-fixed point). Assume that a self map, $T : M \to M$, has an a.f.p (called, p0). In which case, the point Mp0 is "very near" to the point p0. Here, the distance is less than ϵ, that is., $d(Mp0, p0) < \epsilon$. An a.f.p is a point that is nearly located at its respective f.p.

II. PRELIMINARIES

In this section, some notations and basic notions, such as fundamental definitions and lemmas, from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. Boriceanu [2009] Let M be a non-empty set and $b \geq 1$ be a given real number. A function $d : M \times M \longrightarrow R+$ is called a b-metric provided that for all $p, q, r \in M$ satisfies the following conditions.

d(p, q) = 0 iff p = q;

(ii) d(p, q) = d(q, p);

(iii) (p, q) ≤ b[d(p, r) + (r, q)]

The pair (M, d) is called a b-metric space. Immediately from the notion of b-metric space we have the result, every metric space is a b-metric space with b = 1

Example 2.1. Boriceanu [2009] Let $M = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2$

$d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$

Then, $d(p, q) \leq m\ 2\ [d(p, r) + d(r, q)]$ for all $p, q, r \in M$.

if $m > 2$ then the triangle inequality does not hold.

Definition 2.2. Berinde [2006] Let (M, d) be a metric space and $T : M \to M$. Then $p \in M$ is said to be an ϵ-fixed point (approximate fixed point) of T, if for every $\epsilon > 0$ implies

$d(p, T p) < \epsilon$.

Let $F\epsilon(T) = \{ p \in M : d(p, T p) < \epsilon\}$ denotes the set of all ϵ-fixed point of T for a given $\epsilon > 0$.

Definition 2.3. Berinde [2006] Let $T : M \to M$. Then T has the approximate fixed point property (a.f.p.p) if for every $\epsilon > 0$, $F\epsilon(T) \neq \emptyset$. i.e

Lemma 2.1. Berinde [2006] Let A be a closed subset of a metric space (M, d) and

$T : A \to M$ be a compact map. Then T has a fixed point if and only if it has the approximate fixed point property.

Remark 2.1. Berinde [2006] In the following, by $\delta(A)$ f for a set $A \neq \emptyset$ we will understand the diameter of the set A, i.e., $\delta(A) = \sup\{d(p, q) : p, q \in A\}$.

Definition 2.4. Berinde [2006] Let (M, d) be a metric space, $T : M \to M$ a operator and $\epsilon > 0$.

We define the diameter of the set $F\epsilon(T)$,

i.e., $F\epsilon(T) = \sup\{d(p, q) : p, q \in F\epsilon(T)\}$.

Lemma 2.2. Berinde [2006] Let (M, d) be a metric space, $T : M \to M$ an operator and $\epsilon > 0$.

We assume that: (i) $F\epsilon(T) \neq \emptyset$; and

(ii) for all $\theta > 0$,

there exists $\phi(\theta) > 0$

such that $d(p, q) - d(T p, T q) \leq \theta$ implies that $d(p, q) \leq \phi(\theta)$, for all $p, q \in F\epsilon(T)$.

Then; $\delta(F\epsilon(T)) \leq \phi(2\epsilon)$.

Extended b-metric spaces and the related a.f.p.r's

Definition 2.5. Mohsenalhosseini and Saheli [2021] Let (M, d) be a b-metric space with coefficient $b \geq 1$.

A self map $T : M \to M$ is said to be a Mohsenisaheli type contraction if there exists $k1 \in [0, 1, 2\ )$ with $k1(b + 1) < 1$ such that $d(T p, T q) \leq k1[d(p, q) + d(T p, T q)]$, for all $p, q \in M$.

Definition 2.6. Marudai and Bright [2015] Let (M, d) be a b-metric space with coefficient $b \geq 1$.

A selfmap $T : M \to M$ is said to be a B-contraction if there exists $k1, k2, k3 \in [0, 1)$ with

$$k1(b + 1) + bk2 + b(b + 1)k3 < 1$$

such that

$d(T p, T q) \leq k1[d(p, T p) + d(q, T q)] + k2d(p, q) + k3[d(p, T q) + d(q, T p)]$, for all $\in M$.

Definition 2.7. Bianchini [1972] Let (M, d) be a b-metric space.

A selfmap $T : M \to M$ is said to be a Bianchini contraction if there exists $k \in (0, 1)$ with $kb < 1$

such that $d(T p, T q) \leq kB(p, q)$,

where $B(p, q) = \max\ \{d(p, T p), d(q, T q)\}$, for all $p, q \in M$.

III. CONCLUSION

in this journal paper , we prove the approximate fixed point theorems using various types of contraction mappings, including the B-contraction, the Mohseni-Saheli contraction, and their consequences on b-metric spaces.In this paper, some approximate fixed point theorems are established in a b-metric space by utilizing various types of contraction mappings. Further, some approximate fixed point theorems are newly developed for rational type contractions in a b-metric space. It is worth observing that in the limiting case $\epsilon \longrightarrow 0$, all the results established in the present paper produces more restricted approximate fixed points. The notion of approximate fixed points is consequently not less essential than that of exact fixed points. Results that will be given in the future can be demonstrated in a lesser setting to ensure the existence and uniqueness of the approximate fixed points.

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REFERENCES

[1]. A. Bakhtin. The contraction mapping principle in almost metric spaces. Funct. Anal., 30:26–37, 1989.

[2]. S. Banach. Surles operations dans les ensembles abstract et leur application aux equation integrals. Fund. Math, 3:133–181, 1922.

[3]. M. Berinde. Approximate fixed-point theorems. Stud. Univ. ”Babes-bolyai”, Mathematical I, 1:11–25, 2006.

[4]. V. Berinde. Iterative approximation of fixed points. Editura efemeride, Baia Mare, 2002.

[5]. V. Berinde. On the approximation of fixed points of weak contractive mappings. Carpathian J. Math., 19(1):7–22, 2003.

[6]. R. M. T. Bianchini. Su un problema di S. Reich riguardante la teoria dei punti fissi. Bolletino U.M.I.,, 4(5):103–106, 1972.

[7]. M. Boriceanu. Fixed point theory for multivalued generalized contraction on a set with two b-metric spaces. Studia, univ.”Babes-Bolyai”, Mathematica Libertica, 3:1–14, 2009.

[8]. F. Brouwer. The fixed point theory of multiplicative mappings in topological vector spaces. Mathematics Annalen., 177:283–301, 1908.

[9]. S. K. Chatterjea. Fixed point theorems. C.R. Accad.Bulg. Sci., 25:727–730, 1972.