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A Comprehensive Study of the Separation of Variables Technique for Differential Equations

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Abstract: *This paper aims to provide a comprehensive study of the variable separation technique and its applicability to various types of ordinary differential equations and Partial Differential Equations. The separation of variables is a fundamental and widely applicable technique for finding analytical solutions to certain types of differential equations arising in diverse fields such as physics, engineering, and applied mathematics. To obtain unique solutions by using boundary and initial conditionsthis technique has the crucial role. We explore the theoretical framework, illustrate the technique with detailed examples encompassing linear and nonlinear equations, and discuss the limitations and extensions of this method. Furthermore, we examine the connection between separation of variables and the generation of orthogonal functions and Sturm-Liouville theory, highlighting its significance in the broader context of mathematical analysis.*

Keywords: Variable Separation, Ordinary Differential Equation, Partial Differential Equation, Initial Value Problem, Boundary Value Problem

I. INTRODUCTION

Differential equations is a mathematical expressions which describe the relationship between a function and its derivatives and it show up surprisingly often in a number of fields, including physics, biology, chemistry and economics. The rate of change of a functions, or about the several variable smaking impact on the rate of change of a functions [3], then behind this scenes it is likely that there is the hidden differential equations. Among the various techniques developed for solving differential equations, the method of separation of variables holds a prominent position due to its elegance, effectiveness, and wide range of applicability.

The core idea behind separation of variables is to transform a differential equation involving multiple independent variables into a set of simpler ordinary differential equations, each involving only one variable. This is achieved by assuming that the solution can be expressed as a product of functions, where each function depends solely on one of the independent variables [2]. Many laws from physics can be expressed in the form of differential equations. The classic force equals mass times acceleration. Modelling in Differential Equations means that study specific situations tounderstand the nature of the forces function or relationships involved with the aim of translating the situation into the mathematical relationships [1].This paper is intended to familiarize the student with the basic concepts, principles and methods of analysis and its applications, and they are intended for senior undergraduate or beginning graduate students.

II. ORDINARY DIFFERENTIAL EQUATIONS

The ordinary differential equations are the equations which involves one or more derivatives of the function $y = f(x)$. Examples of such differential equations are:

i) $y' = 3$, or more complicated such as ii) $y'' + 12y = 0$or

iii) $(x^2 y''') + e^x y' - 3xy = (x^3 + x)$.

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Differential equations may be classify in a number of ways. The order of the differential equation is the highest order of derivative that appears in that ordinary differential equation. Therefore orders of above mentioned three differential equations are one, two and three respectively.

It is quite often that such modeling ends up with the ordinary differential equations. The main aim of such modeling is to find solutions to such differential equations, and then to study these solutions to provide an understanding of the situation along with giving the prediction of the behavior of solutions [3,5].

In biological sciences, if someone studies populations of small unicelled organisms, and their rate of growth, then it will be easy to run across one of the basic differential equation models, that of an exponential growth. To model unicelledorganisms population growth of a group of e-coli cells in the Petri dish, for example, if we make the assumption that these cells have unlimited resources, space and food for growth, then these cells will reproduce at fairly specific measurable rates. The trick to figuring out how these cell population is growing to understand that the number of new unicells created over any small time intervals are proportional to the number of unicells present at that time. This means that if we look in the Petri dish and count 500 cells at the particular instace, then the number of new unicells being created at that instance should be exactly five times the number of new cells being created if we had looked in the petri dish and only seen 100 cells. Curiously, the simple observation led to population growth studies about humans by Malthus and others in the nineteenth century, based on exactly these same idea of proportional growth [7,8].

Thus, we have simple observation that the rate of change in the population growth at any particular instance is indirect proportional to the number of cells present at that instance. If you now put these observations into the mathematical statements involving the Population function y = P(t), where 't' stands for time, and P(t) stands for the function which gives the population of unicells at time t then you have become the mathematical modeler.

$P'(t) = k\,P(t)$, or, can written equivalently as, $y' = k\,y$

To solve these stated differential equation means finding a solution of the population function y = f(t), such that when the corresponding derivatives are computed and substituted into the ordinary differential equation, then this equation becomes an identity such as '3x = 3x'.

III. PARTIAL DIFFERENTIAL EQUATIONS

The application of separation of variables to PDEs follows a similar principle but involves more independent variables and often leads to a system of ODEs. The separation of variables technique is particularly powerful for solving linear homogeneous PDEs with specific boundary conditions. The classic examples of PDE includes the heat equation and the wave equation in various coordinate systems.

The Heat Equation: Let us consider the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \propto \frac{\partial^2 u}{\partial x^2}$$

Where the function u(x,t) is the temperature at position x and time t, and 'α' is the thermal diffusivity. Assuming the solution of this equation in the form u(x,t)=X(x)T(t), we substitute it into the heat equation we get:

$X(x)T'(t)=\alpha X''(x)T(t)$

Dividing both sides by X(x)T(t) (assuming they are non-zero) gives:

$\alpha T(t)T'(t)=X(x)X''(x)$

The left side of equation depends only on t and the right side of equation depends only on x, so they must be equal to a separation constant, say $-\lambda$:

$\alpha T(t)T'(t)=-\lambda \Longrightarrow T'(t)+\alpha\lambda T(t)=0 \quad X(x)X''(x)=-\lambda \Longrightarrow X''(x)+\lambda X(x)=0$

The solutions to these differential equationss depend on the value of λ [4,6].

The Wave Equation: let us consider the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

assuming the function u(x, t)=X(x)T(t) yields to:

$c^2 T(t)T''(t)=X(x)X''(x)=-\lambda$

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resulting in two ordinary differential equations for T(t) and X(x). The boundary and initial value conditions, determines the values of λi. e. the coefficients of the superposition [8,10].

IV. SOLVING ORDINARY DIFFERENTIAL EQUATIONS

To find the solution of differential equations is an art and a science. So many different varieties of differential equations are studiedthat there is no one sure-fire method that can find the solution to all differential equations exactly i.e. coming up with the closed form for a solution function, such as y = 3x + 5. However, there are, number of numerical techniques that can give approximate solutions to these differential equations to any desired degree of accuracy.

These techniques are used sometimes to answer a specific questions, but often it is knowledge of an exact solution that leads to better the understanding of the situation being described by such differential equations [1].

In the mathematical analysis, we will concentrate on solving ordinary differential equations exactly, and will not use such numerical techniques. If you are interested in seeing some numerical techniques in action then you might consider trying to solve some differential equations using these Mathematical programs. Note that for solving example y′ = 3, we could have simply integrated both sides of equation,which follows from the very basic idea that anytime we are given two things that are equal, and then as long as we do the same thing to one side of an equation as to the other side of equation then this equality still holds. An obvious example of this principle in action, if x = 2, then x + 6 = 2 + 6 and 4x = 8. To solve the differential equation y′ = 3, is quite straightforward, we just have to integrate both sides of the equation:

$$\int y'dx = \int 3dx\,.$$

Fundamental theorem of Calculus then tells us that the integral of the derivative of the function is just the function itself up to a constant, so that $\int y'dx = y + c_1$, and also that $\int 3dx = 3x + c_2$ where we represent different constants by writing c_1 and c_2, to distinguish them from each other. All these subscripted constants can appear odd, but there will be times when keeping good records of new constants that come along while we were solving differential equations and these constant will be especially important [9]. Just think of it as keeping track of "+" or "–" in an equation - yes it can be somewhat annoying at times, but clearly it's critical!

Then finally we have $y = 3x + (c_2 - c_1)$, which we simplify as the $y = 3x + c$, since until an initial condition is given, we donot actually find the value of any of these constants, so we might as well lump them together under one notation.

The extra condition in addition to the differential equation, is often called an initial value condition, if the condition involves the value of the function when x = 0 or t = 0 i. e.any independent variable is labeled for that function in the given equation [2,4]. Sometimes to identify a specific solution to an ordinary differential equation, several initial value conditions needed to be give, not just about the value of the function, y = f(x), when x = 0, but also giving the value of the first derivative of y when x = 0, or even the value of higher order derivatives of y as well.

V. SEPARATION OF VARIABLES TECHNIQUE

The same approach may be applied for the other differential equations also. Since this method worked so nicely in our first example was that the two sides of the differential equation were neatly separated for us to do each of the integrations. We applynow the same thing for the differential equation:

$$(1) \qquad P'(t) = k\,P(t)$$

Now we could try to integrate both sides of equation (1) directly, then we can write

$$(2) \qquad \int P'(t)dt = \int kP(t)dt$$

Although we can easily do the left side integral, and just end up with P(t) + c, integrating the right-hand side leads us in circles, how can we integrate the function like P(t) if we donot know what the function is (finding that function was the

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whole point, after all!). It appears that we need to take a different approach. Since we don't know what P(t) is, let's try isolating everything involving P(t) and its derivatives on one side of the equation:

$$\frac{1}{P(t)}P'(t) = k \tag{3}$$

Now let's try to integrate both sides of the equation (3) with respect to t again, we get:

$$\int \frac{1}{P(t)}P'(t)dt = \int kdt \tag{4}$$

The reason we're in better shape now is that the right-hand side integral is trivial, and the left-hand side integral can be taken care almost directly, with a quick substitution: if $u = P(t)$ then $du = P'(t)dt$ and so (4) becomes

$$\int \frac{1}{u}du = \int kdt \tag{5}$$

so that

$$\ln|u| = \ln|P(t)| = kt + c_1 \tag{6}$$

where we have gone ahead and combined these constants on one side of equation. Solving this equation for the function P(t), we find that

$$P(t) = \pm e^{kt+c_1} = \pm e^{c}e^{kt} = Ce^{kt} \tag{7}$$

Here we have simply lumped all the unknown constants together as single constant C.
The technique of splitting up the given differential equation by taking everything involving the functions and its derivatives on one side and everything remaining on the other side and then finally integrating is called the separation of variables [4,10]. Thistrickis really useful.

One notational shortcut you can use as you go through the separation of variables technique is to write $P'(t)$ as $\frac{dP}{dt}$ in the original differential equation (1) which then becomes

$$\frac{dP}{dt} = kP \tag{8}$$

When you separate these variables, treat the $\frac{dP}{dt}$ as if it were a fraction (you have probably seen this type of thing done before, remember, this is only a notational shortcut, the derivative $\frac{dP}{dt}$ is one whole unit, of course, and not a fraction!) [1]. Thus to separate the variables inequation (8) we get

$$\frac{1}{P}dP = kdt \tag{9}$$

The integration step looks as if it is happening with respect to variable P on the left hand side and with respect to x on the right hand side of the equation:

$$\int \frac{1}{P}dP = \int kdt \tag{10}$$

after integrating these, you are right back to the equation (6) above.

VI. THE ROLE OF BOUNDARY AND INITIAL CONDITIONS

Boundary and initial conditions are indispensable in determining the unique solution of a differential equation obtained through separation of variables. For PDEs, boundary conditions specify the values or derivatives of the solution on the boundaries of the spatial domain, while initial conditions specify the state of the system at a particular time [3].

The boundary conditions play a crucial role in the eigen value problem arising from the separation of variables. They dictate which values of the separation constant λ lead to non-trivial solutions (eigenvalues) and the corresponding spatial functions (eigenfunctions). These eigen functions often form an orthogonal basis for the space of functions satisfying the boundary conditions, which is essential for satisfying the initial condition through Fourier series or generalized Fourier series expansions.

The initial conditions, on the other hand, are used to determine the coefficients in the series solution, ensuring that the solution matches the initial state of the system [7]. Orthogonality of the eigenfunctions allows for the simple calculations of these coefficients through integration.

VII. ADVANTAGES AND LIMITATIONS OF THE TECHNIQUE

The following are the advantages of the separation of variable technique:

- Analytical Solutions: When applicable, the separation of variables technique provides exact analytical solutions, offering a deep understanding of the system's behaviour.
- Transformation to Simpler Problems: It reduces complex PDEs into simpler ODEs, which are often easier to solve.
- Foundation for Further Analysis: The eigenfunctions and eigenvalues obtained through this method form the basis for Fourier series and other spectral methods, which are crucial in many areas of applied mathematics and physics.
- Conceptual Clarity: The method provides a clear physical interpretation in many cases, such as the normal modes of vibration in the wave equation or the spatial and temporal decay in the heat equation [5-7].
- The following are the limitations of the separation of variable technique:
- Applicability to Linear Homogeneous Equations: The standard separation of variables technique is primarily applicable to linear homogeneous differential equations. Non-homogeneous equations often require modifications or other techniques.
- Specific Boundary Conditions: The method is most effective when the boundary conditions are separable, meaning they can be applied independently to each of the separated functions.
- Coordinate System Dependence: The separability of a PDE often depends on the chosen coordinate system. Equations that are separable in Cartesian coordinates may not be separable in polar or cylindrical coordinates, and vice versa.
- Complexity of Solutions: Even when separable, the resulting ODEs may not always have simple analytical solutions. The superposition of infinite series can also lead to complex expressions [6,8].

VIII. CONCLUSION

The separation of variables technique remains a powerful and elegant method for solving a significant class of linear homogeneous differential equations. Its ability to transform complex multi-variable problems into simpler ODEs, coupled with the power of superposition and Fourier analysis, allows for the analytical solution of many physically significant PDEs. This comprehensive study highlights the fundamental principles, diverse applications, and theoretical underpinnings of this elegant and vital tool, underscoring its enduring significance in the field of differential equations.

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