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Inverse Truncated Negative Binomial Rayleigh Distribution and Its Properties

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Abstract: *Rayleigh distribution has many applications in lifetime studies. In this article, a new three parameter lifetime model called Inverse Truncated Negative Binomial Rayleigh (ITNBR) distribution is introduced and its various properties are discussed. This new distribution is obtained by compounding inverse truncated negative binomial and Rayleigh distribution. The shape properties of the probability density function and hazard rate, model identifiability, moments and median of the ITNBR are studied. The unknown parameters of the distribution are estimated using maximum likelihood method. Simulation is carried out to illustrate the performance of maximum likelihood estimates of model parameters.*

Keywords: Inverse Rayleigh distribution, Marshall-Olkin family of distributions, Maximum likelihood, Truncated negative binomial distribution

I. INTRODUCTION

In the last two decades researchers have greater intention toward the inversion of univariate probability models and their applicability under inverse transformation. The inverse distribution is the distribution of the reciprocal of a random variable. Dubey[1] proposed inverted beta distribution, Voda[2] studied inverse Rayleigh distribution, Folks and Chhikara[3] proposed inverse Gaussian distribution, Prakash[4] studied the inverted exponential model, Sharma et al.[5] introduced inverse Lindley distribution, Gharib et al.[6] studied Marshall-Olkin extended inverse Pareto distribution, Al-Fattah et al.[7] introduced inverted Kumaraswamy distribution and Rana and Muhammad[8] introduced Marshall-Olin extended inverted Kumaraswamy distribution.

The inverse rayleigh (IR) distribution is commonly used in statistical analysis of lifetime or response time data from reliability experiments. For the situations in which empirical studies indicate that the hazard function might be unimodal, the IR distribution would be an appropriate model. Initially, Treyer[9] introduced the inverse Rayleigh distribution as a model for analyzing reliability and survival data. The model later underwent further examination by Voda[2], who observed that the lifetime distributions of various experimental units could be closely approximated with the inverse Rayleigh distribution. Additionally, Voda[2] explored its properties and provided a maximum likelihood (ML) estimator for the scale parameter. Gharraph[10] conducted an in-depth analysis of the inverse Rayleigh distribution, deriving five key measures of location: the mean, harmonic mean, geometric mean, mode, and median. Furthermore, Gharraph[10] explored various estimation methods to determine the unknown parameter of thisdistribution. Almarashi et al.[11] propose a two-parameter extension of the inverse Rayleigh distribution, employing the half-logistic transformation to address limitations in modeling moderately right-skewed or near-symmetrical lifetime data. Furthermore, Chiodo et al.[12] introduce the compound inverse Rayleigh distribution as a model tailored for extreme wind speeds, essential in wind power generation and turbine safety evaluation. They provide a practical framework for real-world data analysis, accompanied by a novel Bayesian estimation approach, supported by extensive numerical simulations and robustness assessments.

The addition of parameters has been proved useful in exploring skewness and tail properties, and also for improving the goodness-of-fit of the generated family. Introduction of a scale parameter leads to accelerate life model and taking powers of survival function introduces a parameter that leads to proportional hazards model. Also, the extended distributions have attracted several statisticians to develop new models because the analytical and computational facilities available in programming softwares such as Mathcad, Mapple, MathLab and R can easily tackle the problems involved in computing special functions in these extended distributions.

Marshall and Olkin[13] discussed a method of adding a new parameter to an existing distribution. It includes the baseline distribution as a special case, and gives more flexibility to model various types of data. One of the important properties of this family is that Marshall-Olkin family of distributions possess stability property in the sense that if the method is applied twice, it returns to the same distribution. Also this family satisfies geometric extreme stability property.

Marshall and Olkin [13] started with a parent survival function $\bar{F}(x)$and considered a family of survival functions given by

$$\overline{G}(x) = \frac{\alpha\overline{F}(x)}{F(x) + \alpha\overline{F}(x)}, \quad \alpha > 0 \quad x \in \mathbb{R}. \tag{1}$$

They described the motivation for the family of distributions (1) as follows:

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with survival function $\bar{F}(x)$. Let

$$U_N = \min(X_1, X_2, ..., X_N), \tag{2}$$

where N is the geometric random variable with probability mass function (pmf) $P(N = n) = \alpha(1- \alpha)^{n-1}$, for n =1,2,… and $0 < \alpha < 1$ and independent of X_i's. Then the random variable U_N hasthe survival function given by (1). If $\alpha > 1$ and N is a geometric random variable with pmf of the form $P(N = n) = \frac{1}{\alpha}(1-\frac{1}{\alpha})^{n-1}$, n = 1, 2, . . . then the random variable $V_N = \max(X_1,X_2, . . . X_N)$also has the survival function as (1).

Nadarajah, Jayakumar and Ristic[14] proposed a new generalization of the Marshal-Olkin family of distributions, by replacing the geometric distribution of N in (2), as truncated negative binomial distribution with pmf given by

$$P(N = n) = \frac{\alpha^\theta}{1-\alpha^\theta}\begin{pmatrix} \theta + n - 1 \\ \theta - 1 \end{pmatrix}(1-\alpha)^n, \; n = 1, 2, \ldots,$$

where $\alpha > 0$ and $\theta > 0$.The authors showed that the random minimum, $U_N = \min(X_1, X_2, . . . X_N)$ has the survival function of the form

$$\overline{G}(x; \alpha, \theta) = \frac{\alpha^\theta}{1-\alpha^\theta}[(F(x) + \alpha\overline{F}(x))^{-\theta} - 1]. \tag{3}$$

Note that if α tend to 1, then $\bar{G}(x; \alpha, \theta)$tends to $\bar{F}(x)$. The family of distributions given in (3) is a generalization of Marshall-Olkin family of distributions, in the sense that when θ = 1, (3) reduces to(1).

Sankaran and Jayakumar [15] introduced a family of distributions by replacing the distribution of N in (2) as the discrete Mittag-Leffler distribution, a generalization of geometric distributionwhose probability generating function (pgf) is given by

$$H(s) = \frac{1}{1 + c(1-s)^\beta}, \quad c > 0, 0 < \beta \leq 1.$$

Using truncated discrete Mittag-Leffler distribution, they derived a family of distributions with parameters βand c having survival function

$$\bar{G}(x) = \frac{1 - F^\beta(x)}{1 + cF^\beta(x)}. \tag{4}$$

Note that, the Marshall-Olkin method applied to F^β the exponentiated form of a parent distribution function F, will also gives rise (4). The family of distributions generated using truncated discrete Mittag- Leffler distribution can also be considered as a generalization of Marshall-Olkin family of distributions, since it reduces to Marshall-Olkin family, when β = 1 and c = $\frac{1-\alpha}{\alpha}$.

A non negative integer valued random variable is said to be discrete Linnik distributed, if it has the pgf

$$H(s) = \begin{cases} \left(\frac{1}{1+c(1-s)^\beta}\right)^\theta & \text{for} \quad 0 < \theta < \infty \\ e^{-c(1-s)^\beta} & \text{for} \quad \theta = \infty. \end{cases}$$

Jayakuamar and Sankaran[16] using truncated discrete Linnik family of distributions with parameters β, θ and c have the survival function

$$\bar{G}(x) = \frac{(1+c)^\theta - [1+cF^\beta(x)]^\theta}{[(1+c)^\theta - 1][1+cF^\beta(x)]^\theta}. \quad (5)$$

In (5), when θ = 1 and β ≠ 1, we obtain the survival function of the family of distributions generated using truncated discrete Mittag-Leffler distribution. When β = 1 and θ ≠ 1 in (5), we obtain the survival function of the family of distributions generated using truncated negative binomial distribution in (3). Also when β = 1 and θ = 1 in (5), we obtain the survival function of Marshall-Olkin scheme, in (1).

In this paper, we study inverse truncated negative binomial Rayleigh (ITNBR) distribution. The new proposed distribution is a generalization of Marshall-Olkin extended inverse Rayleigh, Marshall-Olkin extended inverse exponential, inverse Rayleigh and inverse exponential distribution.

The rest of the paper is organized as follows. In Section II, we discuss a family of distributions, namely inverse truncated discrete Linnik G distribution and their sub model inverse family of distributions generated through truncated negative binomial G family of distributions. In particular, we study inverse truncated negative binomial Rayleigh (ITNBR) distribution in Section III. The shape properties of density and hazard function are studied. The model identifiability of the distribution is proved. In Section IV, some structural properties of ITNBR distribution such as moments and quantiles. Method of generation of random variate from ITNBR distribution is also discussed. Estimation of the model parameters by maximum likelihood estimation is performed in Section V. Simulation study is also carried out in order to establish the consistency property of the maximum likelihood estimates of our proposed model.

II. INVERSE FAMILY OF DISTRIBUTIONS GENERATED THROUGH TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION

Let X follows truncated discrete Linnik family of distributions with survival function S(.) and baseline distribution function F(.). Then $Y = \frac{1}{X}$ is an inverse truncated discrete Linnik random variable with cumulative distribution function (cdf) G(x) given by

$$\begin{aligned} G_Y(x) &= P(Y \le x) \\ &= P(\frac{1}{X} \le x) \\ &= P(X \ge \frac{1}{x}) \\ &= S(1/x) \\ &= \frac{(1+c)^\theta - [1+cF^\beta(1/x)]^\theta}{[(1+c)^\theta - 1][1+cF^\beta(1/x)]^\theta}, \quad \beta, \theta, c > 0; \ x > 0 \end{aligned} \quad (6)$$

Hence, we obtain a new family of distributions, which we named as inverse family of distributions generated through discrete Linnik G distribution.

The probability density function (pdf) and the hazard rate function (hrf) of a random variable from the introduced family are respectively

$$g(y, \beta, c, \theta) = \frac{\beta\theta c(1+c)^\theta y^{-2} f(1/y) F^{\beta-1}(1/y)}{[(1+c)^\theta - 1][1+cF^\beta(1/y)]^{\theta+1}}, \quad (7)$$

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and

$$h(y,\beta,c,\theta)=\frac{\beta\theta cF^{\beta-1}(1/y)f(1/y)}{[1+cF^{\beta}(1/y)][(1+cF^{\beta}(1/y))^{\theta}-1]}. \quad (8)$$

When in equation(6), β = 1 and c = $\frac{1-\alpha}{\alpha}$, the cdf reduces to inverse truncated negative binomialG family of distributions. So the cdf, pdf and hrf of inverse truncated negative binomial G familyof distributions are respectively:

$$G(y;1,\alpha,\theta) = \frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[[\alpha+(1-\alpha)F(1/y)]^{-\theta}-1\right], \quad (9)$$

$$g(y;1,\alpha,\theta) = \frac{\alpha^{\theta}(1-\alpha)\theta y^{-2}f(1/y)}{(1-\alpha^{\theta})(\alpha+(1-\alpha)F(1/y))^{\theta+1}}, \quad (10)$$

and

$$h(y;1,\alpha,\theta) = \frac{\alpha^{\theta}(1-\alpha)\theta y^{-2}f(1/y)}{[\alpha+(1-\alpha)F(1/y)]^{\theta}-\alpha^{\theta}}. \quad (11)$$

A new generalization of inverse rayleigh distribution

Now we consider, generalized inverse Rayleigh distribution generated through inverse truncated negative binomial and Rayleigh distribution. Negative binomial is a generalization of the geometric, and Poisson distributions is a limiting particular case. The negative binomial distribution with support over the set of all non-negative integers is also a generalization of the Poisson distribution in the sense that it can deduced as a hierarchical model if X follows Poisson (Δ) with Δ being a gamma random variable.

III. A DISTRIBUTION FUNCTION

Let X follows Rayleigh distribution with parameter λ > 0 having cdf F(x) = 1- $e^{-(\lambda x)^2}$ and pdf f(x) = $2\lambda^2 x e^{-(\lambda x)^2}$. Hence from (9), the cdf of the random variable Y is given by

$$G(y;\alpha,\theta,\lambda)=\frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[\frac{1-[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]^{\theta}}{[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]^{\theta}}\right]. \quad (12)$$

III.B PROBABILITY DENSITY FUNCTION

The pdf of the new distribution is given by

$$g(y;\alpha,\theta,\lambda)=\frac{2\alpha^{\theta}(1-\alpha)\theta\lambda^2 y^{-3}e^{-(\frac{\lambda}{y})^2}}{(1-\alpha^{\theta})[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]^{\theta+1}}. \quad (13)$$

We refer to this new distribution having cdf (12) as inverted truncated negative binomial Rayleigh distribution with parameters α, θ and λ We write it as ITNBR(y; α, θ, λ).

The graph of g(y) for different values of the parameters is given in Figure 1.

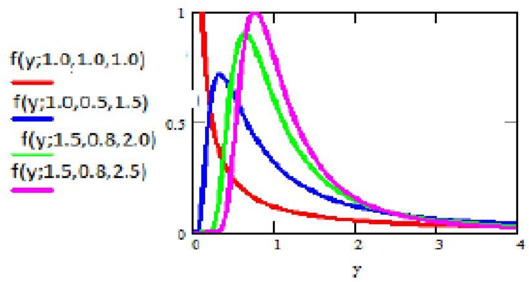

Figure 1: The plots of the pdf of ITNBR distribution.

III. C UNIMODALITY

The pdf of the ITNBR model is either decreasing or unimodal. In order to investigate the critical points of density function, its first derivative with respect to y is

$$g'(y) = \frac{2\alpha^{\theta}(1-\alpha)\theta\lambda^2 y^{-3} e^{-(\frac{\lambda}{y})^2}}{(1-\alpha^{\theta})y[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]^{\theta+1}}\left[-3+2(\frac{\lambda}{y})^2+\frac{2(1-\alpha)(\theta+1)(\frac{\lambda}{y})^2 e^{-(\frac{\lambda}{y})^2}}{[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]}\right].$$

g'(y) = 0 implies,

$$(1-\alpha)e^{-(\frac{\lambda}{y})^2}\left[3+2\theta(\frac{\lambda}{y})^2\right]+2(\frac{\lambda}{y})^2-3=0 \qquad (14)$$

Since equation (14) is a nonlinear equation in y, there may be more than one positive root to (14). If $y = y_0$ is a root of (14), then it corresponds to a local maximum if g'(y) > 0 for all $y < y_0$. It corresponds to a local minimum if g'(y) < 0 for all $y < y_0$ and g'(y) > 0 for all $y > y_0$. It corresponds to a point of inflexion if either g'(y) > 0 for all $y \neq y_0$ or g'(y) < 0 for all y ≠ y0.

III. D HAZARD RATE

The hazard rate is given by

$$h(y) = \frac{2\alpha^{\theta}(1-\alpha)\theta\lambda^2 y^{-3} e^{-(\frac{\lambda}{y})^2}}{[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]\{[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]^{\theta}-\alpha^{\theta}\}}. \qquad (15)$$

The graph of h(y) for different values of the parameters is given in Figure 2.

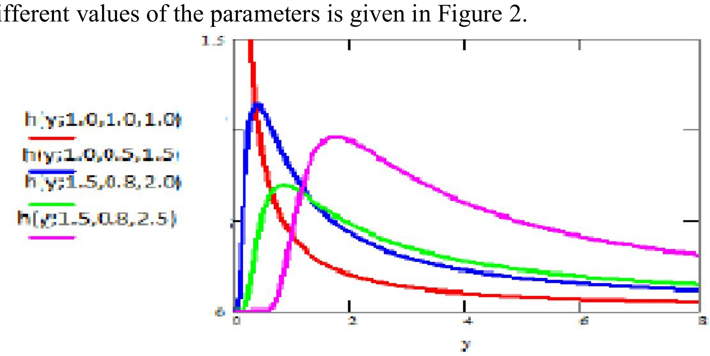

Figure 2: The plots of the hazard rate of ITNBR distribution.

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III.E MODEL IDENTIFIABILITY

We have to prove model identifiability only with respect to the parameters α and θ, since theother parameter λ is from the parent distribution. Let us suppose that $G(y; \alpha_1, \theta_1) = G(y; \alpha_2, \theta_2)$ for all $y > 0$. We will show that this condition implies that $\alpha_1 = \alpha_2$ and $\theta_1 = \theta_2$ For proving model identifiability, we use Theorem 2.4 of Chandra[17].

Proposition : The class of all mixing distribution relative to the ITNBR distribution is identifiable.
Proof : If N_i is truncated negative binomial random variable, truncated at 0, then the probabilitygenerating function is

$$\phi_i(s) = (1-\alpha)^{\theta}\left[\frac{1}{(1-s\alpha)^{\theta}} - 1\right]; \quad i = 1, 2.$$

From the cdf of N_i, we have
$G_1 < G_2$ when $\alpha_1 = \alpha_2$ and $\theta_1 < \theta_2$ and $G_1 < G_2$ when $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$.
Let $D_{\varphi 1}(s) = (-\infty, \alpha_1)$ and $D_{\varphi 2}(s) = (-\infty, \alpha_2)$ and $s = \frac{1}{\alpha 1}$. Hence

$$\lim_{s \to \frac{1}{\alpha_1}} \phi_1(s) = (1-\alpha_1)^{\theta_1}\left[\frac{1}{(1-\frac{1}{\alpha_1}\alpha_1)^{\theta_1}} - 1\right] = \infty. \tag{16}$$

When $\theta_1 = \theta_2$ and $\alpha_1 < \alpha_2$, we obtain

$$\lim_{s \to \frac{1}{\alpha_1}} \phi_2(s) = (1-\alpha_2)^{\theta_1}\left[\frac{1}{(1-\frac{1}{\alpha_1}\alpha_2)^{\theta_1}} - 1\right] > 0.$$

So

$$\lim_{s \to \alpha_1} \frac{\phi_2(s)}{\phi_1(s)} = 0$$

and thus the identifiability is proved. Hence the cdf G is identifiable with respect to α and θ.

III.F EXPANSION FOR DISTRIBUTION FUNCTION AND DENSITY FUNCTION

If $|z| < 1$ and $k > 0$, we have

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k) j!} z^j \tag{17}$$

where $\Gamma(.)$ is the gamma function.
By using (17), the cdf of ITNBR distribution can be expressed as

$$\begin{aligned} G(y) &= \frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]^{-\theta} - 1\right] \\ &= \frac{\alpha^{\theta}}{1-\alpha^{\theta}}\left[\sum_{j=0}^{\infty} \frac{\Gamma(\theta+j)}{\Gamma(\theta) j!}(1-\alpha)^j e^{-j\left(\frac{\lambda}{y}\right)^2} - 1\right]. \end{aligned} \tag{18}$$

In similar manner the pdf of ITNBR distribution can be expressed as

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$$g(y) = \sum_{j=0}^{\infty} \frac{\alpha^\theta}{1-\alpha^\theta} \frac{\Gamma(\theta+j+1)}{\Gamma(\theta+1)j!}(1-\alpha)^{j+1} 2\theta\lambda^2 y^{-3} e^{-(j+1)\left(\frac{\lambda}{y}\right)^2}. \quad (19)$$

IV. GENERAL PROPERTIES OF ITNBR DISTRIBUTION

IV. A MOMENTS

We know that moments are important in any statistical analysis. In this subsection, we presentr^{th} moments of ITNB distribution.

From the definition of moments, we have

$$\begin{aligned} E(Y^r) &= \int_0^{\infty} y^r \sum_{j=0}^{\infty} \frac{\alpha^\theta}{1-\alpha^\theta} \frac{\Gamma(\theta+j+1)}{\Gamma(\theta+1)j!}(1-\alpha)^{j+1} 2\theta\lambda^2 y^{-3} e^{-(j+1)(\frac{\lambda}{y})^2} dy \\ &= \sum_{j=0}^{\infty} \frac{\alpha^\theta}{1-\alpha^\theta} \frac{\Gamma(\theta+j+1)}{\Gamma(\theta+1)j!}(1-\alpha)^{j+1} 2\theta\lambda^2 \int_0^{\infty} y^{r-3} e^{-(j+1)(\frac{\lambda}{y})^2} dy. \end{aligned}$$

Put $x=\lambda^2 (j+1) y^{-2}$. Then

$$\begin{aligned} E(Y^r) &= \sum_{j=0}^{\infty} \frac{\alpha^\theta}{1-\alpha^\theta} \frac{\Gamma(\theta+j+1)}{\Gamma(\theta+1)(j+1)!}\theta(1-\alpha)^{j+1} \lambda^r (j+1)^{\frac{r}{2}} \int_0^{\infty} x^{-\frac{r}{2}} e^{-x} dx \\ &= \sum_{j=0}^{\infty} \frac{\alpha^\theta}{1-\alpha^\theta} \frac{\Gamma(\theta+j+1)}{\Gamma(\theta+1)(j+1)!}\theta(1-\alpha)^{j+1} \lambda^r (j+1)^{\frac{r}{2}} \Gamma(1+\frac{r}{2}), \quad . \quad (20) \end{aligned}$$

By putting r = 1 and r = 2 in (20), we can easily obtain the mean and variance of ITNBR distribution.

IV.B SIMULATION AND QUANTILES

Random variable Y having ITNBR distribution can be easily simulated by inverting the cdf. Let U has unformU(0; 1) distribution, then

$$\frac{\alpha^\theta}{1-\alpha^\theta}\left[[1-(1-\alpha)e^{-(\frac{\lambda}{y})^2}]^{-\theta}-1\right]=U,$$

which yields

$$Y = \lambda\left\{-\log\left[\frac{1}{1-\alpha}\left(1-\left[\frac{\alpha^\theta}{U(1-\alpha^\theta)+\alpha^\theta}\right]^{\frac{1}{\theta}}\right)\right]\right\}^{-\frac{1}{\beta}}. \quad (21)$$

In addition, the q^{th} quantile y_q of ITNBR distribution is given by

$$y_q = \lambda\left\{-\log\left[\frac{1}{1-\alpha}\left(1-\left[\frac{\alpha^\theta}{q(1-\alpha^\theta)+\alpha^\theta}\right]^{\frac{1}{\theta}}\right)\right]\right\}^{-\frac{1}{2}}. \quad (22)$$

$0 < q < 1$.

In particular, the median of ITNBR distribution on quantiles:

$$\text{Median} = \lambda\left\{-\log\left[\frac{1}{1-\alpha}\left(1-\left[\frac{\alpha^{\theta}}{0.5(1-\alpha^{\theta})+\alpha^{\theta}}\right]^{\frac{1}{\theta}}\right)\right]\right\}^{-\frac{1}{2}}. \quad (23)$$

The Bowley's Skewness based on quantiles :

$$S = \frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},$$

and the Moors' Kurtosis based on octiles:

$$K = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)},$$

where Q(.) represents the quantile function of X. These measures are less sensitive to outliers and they exist even for distributions without moments. Skewness measures the degree of the long tail and kurtosis is a measure of the degree of peakedness. When the distribution is symmetric, S = 0 and when the distribution is left(or right) skewed, S < 0 (or S > 0). As K increases, the tail of the distribution becomes heavier.

V. ESTIMATION OF THE PARAMETERS

V.A MAXIMUM LIKELIHOOD ESTIMATION

Several approaches for parameter estimation have been proposed in the literature, but maximum likelihood method is the most commonly employed. We consider estimation of the unknown parameters of ITNB distribution by the method of maximum likelihood. Let $y_1, y_2, \ldots y_n$ be observed values from the ITNB distribution with parameters α, θ and λ. The log-likelihood function for (α, θ, λ)is given by

$$\log L = n\log\left[\frac{\alpha^{\theta}(1-\alpha)2\theta\lambda^2}{1-\alpha^{\theta}}\right] - 3\sum_{i=1}^{n}\log(y_i) - \sum_{i=1}^{n}\left(\frac{\lambda}{y_i}\right)^2 - (\theta+1)\sum_{i=1}^{n}\log[1-(1-\alpha)e^{-(\frac{\lambda}{y_i})^2}].$$

The derivatives of the log-likelihood function with respect to the parametersα, θ and λ are givenby respectively,

$$\frac{\partial\log L}{\partial\alpha} = \frac{n\theta}{\alpha} - \frac{n}{1-\alpha} + \frac{n\theta\alpha^{\theta-1}}{1-\alpha^{\theta}} - \sum_{i=1}^{n}\frac{(\theta+1)e^{-(\frac{\lambda}{y_i})^2}}{1-(1-\alpha)e^{-(\frac{\lambda}{y_i})^2}} \quad (24)$$

$$\frac{\partial\log L}{\partial\theta} = n\log(\alpha) - \frac{n}{\theta} + \frac{n\alpha^{\theta}\log(\alpha)}{1-\alpha^{\theta}} - \sum_{i=1}^{n}\log[1-(1-\alpha)e^{\;(\frac{\lambda}{y_i})^2}] \quad (25)$$

$$\frac{\partial\log L}{\partial\lambda} = \frac{2n}{\lambda} - \frac{2}{\lambda}\sum_{i=1}^{n}\left(\frac{\lambda}{y_i}\right)^2 - \sum_{i=1}^{n}\frac{(\theta+1)(1-\alpha)\frac{2}{\lambda}(\frac{\lambda}{y_i})^2 e^{-(\frac{\lambda}{y_i})^2}}{1-(1-\alpha)e^{-(\frac{\lambda}{y_i})^2}} \quad (26)$$

The maximum likelihood estimates of (α, θ, λ), say $(\hat{\alpha},\hat{\theta},\hat{\lambda})$ are the simultaneous solutions ofthe equation $\frac{\partial\log L}{\partial\alpha}=0$, $\frac{\partial\log L}{\partial\theta}=0$ and $\frac{\partial\log L}{\partial\lambda}=0$ .Maximization of the likelihood function canbe performed by using nlm or optim in **R** Statistical package.

The normal approximation of the maximum likelihood estimates of the parameters can beadopted for constructing approximate confidence intervals and for testing hypotheses on the parameters (α, θ, λ). Under conditions that are fulfilled for the parameters in the interior of the parameterspace and applying the usual large sample approximation, it can be shown that$\sqrt{n}(\emptyset-\widehat{\emptyset})$ canbe approximated by a multivariate normal distribution with zero means and variance-covariancematrix $\mathbf{K}^{-1(}(\emptyset)$ where $\mathbf{K}(\emptyset)$ is the unit expected information matrix.

As n tends to infinity, we have the asymptotic result

$\mathbf{K}(\emptyset) = \lim_{n\to\infty} \frac{1}{n}\, I\,(\emptyset)$

where$I\,(\emptyset)$is the observed Fisher information matrix. Since $\mathbf{K}(\emptyset)$ involves the unknown parameters of $\emptyset$, we may replace it with the MLE $\widehat{\emptyset}$. Thus, the average matrix estimated at$\widehat{\emptyset}$, say $I\,(\emptyset)$, can be used to estimate $\mathbf{K}(\emptyset)$. The estimated multivariate normal distribution can thus be used to construct approximate confidence intervals for the unknown parameters and for the hazard rateand survival function.

V.B SIMULATION

We asses the performance of the maximum likelihood estimates of ITNBR (α, θ, λ) distribution by conducting simulation for different sample sizes and parameter values. We use equation (21) to generate random samples from the ITNBR distribution with parameters α, θ and λ. The different sample sizes considered in the simulation are n = 30; 70; 100 and 200. We have used nlm package in **R** software to find the estimate. We have repeated the process 1000 times and report the average estimates and associated mean square errors in Table 1.

n		$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\hat{\theta}$	MSE($\hat{\theta}$)	$\hat{\lambda}$	MSE($\hat{\lambda}$)
30	α = 0.5	0.599	0.295	0.573	1.056	1.234	0.267
70	θ = 0.5	0.483	0.028	0.580	0.841	1.056	0.145
100	λ = 1.0	0.563	0.016	0.432	0.184	0.984	0.127
200		0.541	0.012	0.506	0.071	1.051	0.112
30	α = 1.5	1.539	0.295	0.487	0.267	1.341	0.243
70	θ = 0.5	1.578	0.244	0.522	0.165	0.896	1.420
100	λ = 1.0	1.560	0.196	0.476	0.093	1.254	0.937
200		1.553	0.144	0.523	0.051	1.187	0.532
30	α = 0.5	0.829	4.329	1.554	0.507	0.831	2.354
70	θ = 1.5	0.432	3.012	1.468	0.302	0.776	1.423
100	λ = 1.0	0.306	2.019	1.547	0.187	0.813	0.937
200		0.458	1.282	1.459	0.103	0.913	0.532
30	α = 5.0	6.265	0.304	1.732	0.159	1.541	1.056
70	θ = 1.5	5.661	0.109	1.624	0.127	1.320	0.841
100	λ = 1.0	4.847	0.085	1.485	0.098	1.023	0.821
200		5.062	0.021	1.526	0.074	0.994	0.563

Table 1 : Simulation results for different values of the parameters α, θ and λ.

From Table 1, we can see that as the sample size increase, the estimated values are close to theactual values and the mean square errors decreases, which establishes the consistency property ofthe MLEs.

VI. CONCLUSION

We discussed a new family of inverse distribution namely Inverse truncated negative binomial family of distribution. A particular member of the family, three parameter inverse truncated negative binomial Rayleigh distribution studied in detail. The density function can be expressed as compact form. The explicit expression for the ordinary moments and Quantiles are derived. We discuss the maximum likelihood estimation of the model parameters and simulation for different sample sizes and parameter values

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REFERENCES

[1] Dubey, S.D. Compound gamma, beta and F distribution. Metrika, 16, 27-31, 1970

[2]VodaV.G. On the inverse Rayleigh distributed random variable. Reports of Statistical Application Research, 19, 13-21, 1972.

[3] Folks, J., Chhikara, R.. The inverse Gaussian distribution and its statistical application-a review. Journal of Royal Statistical Society
Series B, 40, 263-289, 1978.

[4] Prakash, G. Inverted exponential distribution under a Bayesian view point. Journalof Modern Applied Statistical Methods, 11, 190-202, 2012.

[5]Sharma, V.K., Singh, S.K., Singh, U., Agiwal, V.. The inverse Lindley distribution:a stress-strength reliability model with application to
head and neck cancer data. JournalIndustrial and Production Engineering, 32, 162-173, 2015.

[6] Gharib, M., Mohammed, B.I., Aghel, W.E.R. Marshall-Olkin extended inverse Paretodistribution and its application. International
Journal of Statistics and Probability, 6, 71-84, 2017.

[7] Al-Fattah, A., El-Helbawy, A., Al-Dayian, G. Inverted Kumaraswamy distribution :properties and estimation. Pakistan Journal of Statistics, 33, 37-61, 2017

[8]Rana, M.U., Muhammed Ahsan ulHaq. . The Marshall-Olkin extended invertedKumaraswamy distribution: Theory and application Journal of King Saud University, 2018.https://doi.org/10.1016/j.jksus.2018.05.021.

[9] Treyer, V. N. Inverse Rayleigh (IR) model. Proceedings USSR Academic Scinece.,1964.

[10] Gharraph, M.K.. Comparison of estimators of location measures of an inverse Rayleighdistribution. Egyptian Statistical Journal, 37, 295-309, 1993.

[11] Almarshi, A. M., Badr, M. M., Jamal, F., Chesneau, C. Statistical inference of thehalf-logistic inverse Rayleigh distribution. Entropy, 22, 449, 2020.

[12] Chiodo, E, Fantauzzi, M., Mazzanti, G. The compound inverse Rayleigh as anextreme wind speed distribution and its bayes estimation. Energies, 15, 861, 2022.

[13]Marshall, A.W., Olkin, I. A new method for adding a parameter to a family ofdistributions with application to the exponential and Weibull families. Biometrika, 84, 641-652, 1997.

[14] Nadarajah, S., Jayakumar, K., Risti_c, M.M. A new family of lifetime models. Journalof Statistical Computation and Simulation, 83, 1389-1404, 2013.

[15] Sankaran, K.K., Jayakumar, K. A new extended uniform distribution. InternationalJournal of Statistical Distributions and Applications, 2, 35-41, 2016.

[16] Jayakumar, K., Sankaran, K.K. Discrete Linnik Weibull distribution. Communications in Statistics - Simulation and Computation 48,3092-3117, 2019

[17] Chandra, S. On the mixtures of probability distributions. Scandinavian Journal ofStatistics, 4, 105-112, 1977.