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On Trirecurrence Property for Wely's Curvature Tensor in Finsler Space

Abdalstar A. Saleem[1], Alaa A. Abdallah[*2]

Department of Mathematics, Faculty of Sciences, Aden University, Aden, Yemen[1]

Department of Mathematics, Education Faculty, University of Abyan, Yemen[1]

[2]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India[2]

Abstract: *This paper introduces a Finsler space which Wely's curvature tensor W^i_{jkh} satisfies the trirecurrence property in sense of Cartan. Further, we study the relations between the Wely's curvature tensor W^i_{jkh}, normal projective curvature tensor N^i_{jkh} and Berwald curvature tenser H^i_{jkh}.*

Keywords: $W-$ trirecurrent Finsler space, Wely's curvature tensor W^i_{jkh}, Affinely connected space

I. INTRODUCTION

The recurrent and birecurrent Finsler spaces have been studied by [10, 18, 19, 20, 21, 22]. The trirecurrent Finsler spaces for different curvature tensors have been studied by [6, 7, 8, 9, 11, 15, 17]. An affinely connected space for $hv-$curvature tensor that satisfy the birecurrence property discussed by [2, 12].

An $n-$dimensional Finsler space F_n equipped with the line elements (x, y) and the fundamental metric function F that positive homogeneous of degree one in y^i [1, 3, 13, 14]. The vectors y_i and y^i satisfy

a) $y_i y^i = F^2$, b) $\dot{\partial}_i y_j = \dot{\partial}_j y_i = g_{ij}$, c) $g_{it} y^i = y_t$ and d) $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2$.

Cartan's covariant derivative of the fundamental metric function F, vector y^i and unit vector l^i vanish identically, i.e.

a) $F_{|l} = 0$, b) $y^i_{|l} = 0$, and c) $g_{jk|l} = 0$,

Cartan's covariant derivative of an arbitrary tensor T^i_h with respect to x^l is given by [5]

a) $\dot{\partial}_j(T^i_{h|l}) - (\dot{\partial}_j T^i_h)_{|l} = T^r_h(\dot{\partial}_j \Gamma^{*i}_{lr}) - T^i_r(\dot{\partial}_j \Gamma^{*r}_{lj}) - (\dot{\partial}_r T^i_h) P^r_{jl}$,

where b) $P^r_{jl} = (\dot{\partial}_j \Gamma^{*r}_{hl}) y^h$ and c) $P^i_{jl} = g^{ih} P_{hjl}$.

The Berwald curvature tensor H^i_{jkh} is positively homogeneous of degree zero in y^i and skew-symmetric in its last two lower indices which defined by [16]

$H^i_{jkh} = \partial_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^i_{rk} G^r_{jh} - h/k.$

In view of Euler's theorem on homogeneous functions, we have the following relations [4]

(1.4) a) $\dot{\partial}_j H^i_{kh} = H^i_{jkh}$, b) $H^i_{jkh} y^j = H^i_{kh}$, c) $H_{ijkh} = g_{jr} H^r_{ikh}$,

d) $H^i_{kh} y^k = H^i_h$, e) $H^i_{kh} = \dot{\partial}_k H^i_h$, f) $H_{jk} = H^r_{jkr}$,

g) $H_k = H^r_{kr}$, h) $H = \frac{1}{n-1} H^r_r$ and l) $H^r_{rkh} = H_{kh} - H_{hk}$.

The relation between the normal projective curvature tensor N^i_{jkh} and Berwald curvature tenser H^i_{jkh} satisfies [6]

(1.5) $N^i_{jkh} = H^i_{jkh} - \frac{1}{n+1} y^i \dot{\partial}_j H^r_{rkh}$,

where the normal projective curvature tensor N^i_{jkh} is homogeneous of degree zero in y^i.

Contracting the indices i and j in (1.5) and using the fact that the tensor H^r_{rkh} is positively homogeneous of degree zero in y^i, we get

(1.6) $N^r_{rkh} = H^r_{rkh}$.

Transvecting (1.5) by y^j and using (1.4b), we get

(1.7) $N^i_{jkh} y^j = H^i_{kh}$.

The Wely's curvature tensor W^i_{jkh} and normal projective curvature tensor N^i_{jkh} are connected by [16]

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(1.8) a) $W^i_{jkh} = N^i_{jkh} + 2(\delta^i_k M_{hj} - M_{kh}\delta^i_j - k|h)$,

where b) $M_{jk} = -\frac{1}{n^2-1}\left(nN_{jk} + N_{kj}\right)$ and c) $N_{jk} = N^r_{jkr}$.

The Wely's curvature tensor W^i_{jkh} satisfies the following [16]

(1.9) a) $W^i_{jkh}y^j = W^i_{kh}$, b) $W^i_{kh}y^k = W^i_h$ and c) $W^i_h y^h = 0$.

A Finsler space whose connection parameter G^i_{jk} is independent of y^i is called an *affinely connected space* [16]. Thus, one of the equivalent equations characterizes an affinely connected space

(1.10) a) $G^i_{jkh} = 0$ and b) $C_{ijk|h} = 0$.

The connection parameters of Cartan Γ^{*i}_{kh} and Berwald G^i_{jk} coincide in affinely connected space and they are independent of the direction argument, i.e.

(1.11) a) $\dot\partial_j G^i_{kh} = 0$ and b) $\dot\partial_j \Gamma^{*i}_{kh} = 0$.

Cartan's connection parameter Γ^{*i}_{kh} coincides with Berwald's connection parameter G^i_{kh} for a Landsberg space, which is characterized by [16]

(1.12) $y_r G^r_{jkh} = -2C_{jkh|r}y^r = -2P_{jkh} = 0$.

The W −recurrent Finsler space and W −birecurrent Finsler space introduced and characterized by [21, 22]

(1.13) $W^i_{jkh|l} = \lambda_l W^i_{jkh}$, $W^i_{jkh} \neq 0$

(1.14) $W^i_{jkh|l|m} = a_{lm} W^i_{jkh}$,

where λ_l and a_{lm} are non-zero covariant vector field and non-zero covariant tensor field of second order, respectively.

II. MAIN RESULTS

Definition 2.1. Finsler space F_n which the Wely's curvature tensor W^i_{jkh} satisfies the trirecurrence property i.e. characterized by

(2.1) $W^i_{jkh|l|m|n} = a_{lmn} W^i_{jkh}$, $W^i_{jkh} \neq 0$.

where a_{lmn} is non-zero covariant tensor field of third order. This space will be called a W – *trirecurrent Finsler space*. And denote it briefly by $WTR - F_n$.

Transvecting (2.1) by y^j, using (1.9a) and (1.2b), we get

(2.2) $W^i_{kh|l|m|n} = a_{lmn} W^i_{kh}$.

Transvecting (2.2) by y^k , using (1.9b) and (1.2b), we get

(2.3) $W^i_{h|l|m|n} = a_{lmn} W^i_h$.

Thus, we conclude

Theorem 2.1. *In $WTR - F_n$, the Wely's torsion tensor W^i_{jk} and Wely's deviation W^i_h are trirecurrent.*

Differentiating (1.8a) covariantly with respect to x^l, x^m and x^n in sense of Cartan, we get

(2.4) $N^i_{jkh|l|m|n} = W^i_{jkh|l|m|n} + 2(\delta^i_j M_{kh|l|m|n} + \delta^i_h M_{jk|l|m|n})$.

Using (2.1) and (1.8a) in (2.4), we get

$N^i_{jkh|l|m|n} = a_{lmn}[N^i_{jkh} - 2\left(\delta^i_j M_{kh} + \delta^i_h M_{jk}\right)] + 2(\delta^i_j M_{kh|l|m|n} + \delta^i_h M_{jk|l|m|n})$.

Contracting i and h in above equation and using (1.8c) and the property skew –symmetric for M_{jk}, we get

$N_{jk|l|m|n} = a_{lmn}[N_{jk} - 2(1-n)M_{jk}] + 2(1-n)M_{jk|l|m|n}$.

Using (1.8b) in above equation, we get

$N_{jk|l|m|n} = a_{lmn}N_{jk} - \frac{2}{n+1}a_{lmn}\left(nN_{jk} + N_{kj}\right) + \frac{2}{n+1}(nN_{jk|l|m|n} + N_{kj|l|m|n})$.

Using the property skew – symmetric for N_{jk} in above equation, we get

$$N_{jk|l|m|n} = a_{lmn}N_{jk} - 2a_{lmn}N_{jk} + 2N_{jk|l|m|n}$$

which can be written by

(2.5) $N_{jk|l|m|n} = a_{lmn}N_{jk}$.

Thus, we conclude

Theorem 2.2. *In $WTR - F_n$, if M_{jk} and N_{jk} are property skew – symmetric, then N_{jk} is trirecurrent.*

Differentiating (1.8b) covariantly with respect to x^l , x^m and x^n in the sense of Cartan, using (2.5), we get

(2.6) $M_{jk|l|m|n} = -\frac{1}{n^2-1} a_{lmn}(nN_{jk} + N_{kj})$.

Using (1.8b) in (2.6), we get

(2.7) $M_{jk|l|m|n} = a_{lmn} M_{jk}$.

Using (2.1) and (2.6) in (2.7), we get

(2.8) $N^i_{jkh|l|m|n} = a_{lmn} N^i_{jkh}$.

Thus, we conclude

Theorem 2.3. *In $WBR - F_n$, the tensor M_{jk} and normal projective curvature tensor N^i_{jkh} are trirecurrent.*

Contracting the indies i and j in (2.8) and using (1.6), we get

(2.9) $H^r_{rkh|l|m|n} = a_{lmn} H^r_{rkh}$.

Transvecting (2.8) by y^j, using (1.7) and (1.2b), we get

(2.10) $H^i_{kh|l|m|n} = a_{lmn} H^i_{kh}$.

Transvecting (2.10) by y^k, using (1.4d) and (1.2b), we get

(2.11) $H^i_{h|l|m|n} = a_{lmn} H^i_h$.

Contracting the indies i and h in (2.10) and using (1.4g), we get

(2.12) $H_{k|l|m|n} = a_{lmn} H_k$.

Contracting the indies i and h in (2.11) and using (1.4h), we get

(2.13) $H_{|l|m|n} = a_{lmn} H$.

From above equations, we conclude

Theorem 2.4. *In $WTR - F_n$, the curvature tensor H^r_{rkh}, torsion tensor H^i_{kh}, deviation tensor H^i_h, curvature vector H_k and scalar curvature H are trirecurrent.*

In next results, we obtain thenecessary and sufficient condition for some tensors to be trirecurrent in $WTR - F_n$. Differentiating (2.10) partially with respect to y^j, we get

(2.14) $\dot{\partial}_j(H^i_{kh|l|m|n}) = (\dot{\partial}_j a_{lmn}) H^i_{kh} + a_{lmn} \dot{\partial}_j H^i_{kh}$.

Using commutation formula exhibited by (1.3a) for H^i_{kh} in (2.14), using (1.4a), we get

$$
\begin{aligned}
(2.15)\ & H^i_{jkh|l|m|n} + [\{H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - H^i_{rkh}P^r_{jl}\}_{|m} + H^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|l}(\dot{\partial}_j\Gamma^{*r}_{km}) \\
& - H^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl}) - H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) \\
& - H^i_{skh}P^s_{rl}\}P^r_{jm}]_{|n} + H^r_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) - H^i_{rh|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) \\
& - H^i_{kh|l|r}(\dot{\partial}_j\Gamma^{*r}_{mn}) \\
& - [\{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl}) - H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{skh}P^s_{rl}\}_{|m} \\
& + H^s_{kh|l}(\dot{\partial}_r\Gamma^{*i}_{sm}) - H^i_{sh|l}(\dot{\partial}_r\Gamma^{*s}_{km}) - H^i_{ks|l}(\dot{\partial}_r\Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_r\Gamma^{*s}_{lm}) \\
& - \{H^i_{skh|l} + H^t_{kh}(\dot{\partial}_s\Gamma^{*i}_{tl}) - H^i_{th}(\dot{\partial}_s\Gamma^{*t}_{kl}) - H^i_{kt}(\dot{\partial}_s\Gamma^{*t}_{hl}) - P^t_{rl}\}]P^r_{jn} = (\dot{\partial}_j a_{lmn}) H^i_{kh} + a_{lmn} H^i_{jkh}.
\end{aligned}
$$

This shows that

(2.16) $H^i_{jkh|l|m|n} = a_{lmn} H^i_{jkh}$

if and only if

$$
\begin{aligned}
(2.17)\ & [\{H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - H^i_{rkh}P^r_{jl}\}_{|m} + H^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|l}(\dot{\partial}_j\Gamma^{*r}_{km}) \\
& - H^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl}) - H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) \\
& - H^i_{skh}P^s_{rl}\}P^r_{jm}]_{|n} + H^r_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) - H^i_{rh|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) \\
& - H^i_{kh|l|r}(\dot{\partial}_j\Gamma^{*r}_{mn}) \\
& - [\{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl}) - H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{skh}P^s_{rl}\}_{|m}
\end{aligned}
$$

$$+H^s_{kh|l}(\dot{\partial}_r\Gamma^{*i}_{sm}) - H^i_{sh|l}(\dot{\partial}_r\Gamma^{*s}_{km}) - H^i_{ks|l}(\dot{\partial}_r\Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_r\Gamma^{*s}_{lm})$$
$$-\{H^i_{skh|l} + H^t_{kh}(\dot{\partial}_s\Gamma^{*i}_{tl}) - H^i_{th}(\dot{\partial}_s\Gamma^{*t}_{kl}) - H^i_{kt}(\dot{\partial}_s\Gamma^{*t}_{hl}) - P^t_{rl}\}]P^r_{jn} - (\dot{\partial}_j a_{lmn})H^i_{kh} = 0.$$

Thus, we conclude

Theorem 2.5. *In $WTR - F_n$, the Berwald curvature tensor H^i_{jkh} is trirecurrent if and only if (2.17) holds.*

Transvecting (2.15) by g_{ti}, using (1.4c) and (1.2c), we get

$$H_{jtkh|l|m} + g_{ti}[\{H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - H^i_{rkh}P^r_{jl}\}_{|m} + H^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm}))2.18(-H^i_{rh|l}(\dot{\partial}_j\Gamma^{*r}_{km}) -$$
$$H^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl})$$
$$-H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{skh}P^s_{rl}\}P^r_{jm}]_{|n} + g_{ti}[H^r_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) - H^i_{rh|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn})$$
$$-H^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - H^i_{kh|l|r}(\dot{\partial}_j\Gamma^{*r}_{mn}) - \{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl})$$
$$-H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{skh}P^s_{rl}\}_{|m} + H^s_{kh|l}(\dot{\partial}_r\Gamma^{*i}_{sm}) - H^i_{sh|l}(\dot{\partial}_r\Gamma^{*s}_{km})$$
$$-H^i_{ks|l}(\dot{\partial}_r\Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_r\Gamma^{*s}_{lm}) - \{H^i_{skh|l} + H^t_{kh}(\dot{\partial}_s\Gamma^{*i}_{tl}) - H^i_{th}(\dot{\partial}_s\Gamma^{*t}_{kl}) - H^i_{kt}(\dot{\partial}_s\Gamma^{*t}_{hl})$$
$$-H^i_{tkh}P^t_{rl}\}]P^r_{jn} = g_{ti}(\dot{\partial}_j a_{lmn})H^i_{kh} + a_{lmn}H_{jtkh}.$$

This shows that

(2.19) $\quad H_{jtkh|l|m|n} = a_{lmn}H_{jtkh}$

if and only if

(2.20) $$g_{ti}[\{H^r_{kh}(\dot{\partial}_j\Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j\Gamma^{*r}_{hl}) - H^i_{rkh}P^r_{jl}\}_{|m} + H^r_{kh|l}(\dot{\partial}_j\Gamma^{*i}_{rm})$$
$$-H^i_{rh|l}(\dot{\partial}_j\Gamma^{*r}_{km}) - H^i_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl})$$
$$-H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{skh}P^s_{rl}\}P^r_{jm}]_{|n} + g_{ti}[H^r_{kh|l|m}(\dot{\partial}_j\Gamma^{*i}_{rn}) - H^i_{rh|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn})$$
$$-H^i_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{hn}) - H^i_{kh|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - H^i_{kh|l|r}(\dot{\partial}_j\Gamma^{*r}_{mn}) - \{H^i_{rkh|l} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{sl})$$
$$-H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{ks}(\dot{\partial}_r\Gamma^{*s}_{hl}) - H^i_{skh}P^s_{rl}\}_{|m} + H^s_{kh|l}(\dot{\partial}_r\Gamma^{*i}_{sm}) - H^i_{sh|l}(\dot{\partial}_r\Gamma^{*s}_{km})$$
$$-H^i_{ks|l}(\dot{\partial}_r\Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_r\Gamma^{*s}_{lm}) - \{H^i_{skh|l} + H^t_{kh}(\dot{\partial}_s\Gamma^{*i}_{tl}) - H^i_{th}(\dot{\partial}_s\Gamma^{*t}_{kl}) - H^i_{kt}(\dot{\partial}_s\Gamma^{*t}_{hl})$$
$$=0. \quad -H^i_{tkh}P^t_{rl}\}]P^r_{jn} - g_{ti}(\dot{\partial}_j a_{lmn})H^i_{kh}$$

Thus, we conclude

Theorem 2.6. *In $WTR - F_n$, the associate tensor H_{jtkh} of the curvature tensor H^i_{jkh} is trirecurrent if and only if (2.20) holds.*

Contracting the indices i and h in (2.15), using (1.4f) and (1.4g), we get

(2.21) $$H_{jk|l|m|n} + [\{H^r_{kb}(\dot{\partial}_j\Gamma^{*b}_{rl}) - H_r(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^b_{kr}(\dot{\partial}_j\Gamma^{*r}_{bl}) - H_{rk}P^r_{jl}\}_{|m} + (\dot{\partial}_j\Gamma^{*b}_{rm}) - H_{r|l}(\dot{\partial}_j\Gamma^{*r}_{km})$$
$$-H^b_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{bm}) - H_{k|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \{H_{rk|l} + H^s_{kb}(\dot{\partial}_r\Gamma^{*b}_{sl}) - H_s(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^b_{ks}(\dot{\partial}_r\Gamma^{*s}_{bl}) - H_{sk}P^s_{rl}\}P^r_{jm}]_{|n} + H^r_{kb|l|m}(\dot{\partial}_j\Gamma^{*b}_{rn})$$
$$- H_{r|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H^b_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{bn}) - H_{k|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - H_{k|l|r}(\dot{\partial}_j\Gamma^{*r}_{mn})$$
$$-[\{H_{rk|l} + H^s_{kb}(\dot{\partial}_r\Gamma^{*b}_{sl}) - H_s(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^b_{ks}(\dot{\partial}_r\Gamma^{*s}_{bl}) - H^b_{ks}(\dot{\partial}_r\Gamma^{*s}_{bl}) - H_{sk}P^s_{rl}\}_{|m}$$
$$+H^s_{kb|l}(\dot{\partial}_r\Gamma^{*b}_{sm}) - H_{s|l}(\dot{\partial}_r\Gamma^{*s}_{km}) - H^b_{ks|l}(\dot{\partial}_r\Gamma^{*s}_{bm}) - H_{k|s}(\dot{\partial}_r\Gamma^{*s}_{lm})$$
$$-\{H_{sk|l} + H^t_{kb}(\dot{\partial}_s\Gamma^{*b}_{tl}) - H_t(\dot{\partial}_s\Gamma^{*t}_{kl}) - H^b_{kt}(\dot{\partial}_s\Gamma^{*t}_{bl}) - H_{tk}P^t_{rl}\}]P^r_{jn} = (\dot{\partial}_j a_{lmn})H_k + a_{lmn}H_{jk}.$$

This shows that

$$H_{jk|l|m|n} = a_{lmn}H_{jk}$$

if and only if

(2.22) $$[\{H^r_{kb}(\dot{\partial}_j\Gamma^{*b}_{rl}) - H_r(\dot{\partial}_j\Gamma^{*r}_{kl}) - H^b_{kr}(\dot{\partial}_j\Gamma^{*r}_{bl}) - H_{rk}P^r_{jl}\}_{|m} + (\dot{\partial}_j\Gamma^{*b}_{rm}) - H_{r|l}(\dot{\partial}_j\Gamma^{*r}_{km})$$
$$-H^b_{kr|l}(\dot{\partial}_j\Gamma^{*r}_{bm}) - H_{k|r}(\dot{\partial}_j\Gamma^{*r}_{lm}) - \{H_{rk|l} + H^s_{kb}(\dot{\partial}_r\Gamma^{*b}_{sl}) - H_s(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^b_{ks}(\dot{\partial}_r\Gamma^{*s}_{bl}) - H_{sk}P^s_{rl}\}P^r_{jm}]_{|n} + H^r_{kb|l|m}(\dot{\partial}_j\Gamma^{*b}_{rn})$$
$$- H_{r|l|m}(\dot{\partial}_j\Gamma^{*r}_{kn}) - H^b_{kr|l|m}(\dot{\partial}_j\Gamma^{*r}_{bn}) - H_{k|r|m}(\dot{\partial}_j\Gamma^{*r}_{ln}) - H_{k|l|r}(\dot{\partial}_j\Gamma^{*r}_{mn})$$
$$-[\{H_{rk|l} + H^s_{kb}(\dot{\partial}_r\Gamma^{*b}_{sl}) - H_s(\dot{\partial}_r\Gamma^{*s}_{kl}) - H^b_{ks}(\dot{\partial}_r\Gamma^{*s}_{bl}) - H^b_{ks}(\dot{\partial}_r\Gamma^{*s}_{bl}) - H_{sk}P^s_{rl}\}_{|m}$$
$$+H^s_{kb|l}(\dot{\partial}_r\Gamma^{*b}_{sm}) - H_{s|l}(\dot{\partial}_r\Gamma^{*s}_{km}) - H^b_{ks|l}(\dot{\partial}_r\Gamma^{*s}_{bm}) - H_{k|s}(\dot{\partial}_r\Gamma^{*s}_{lm})$$

$$-\{H_{sk|l} + H^t_{kb}(\dot{\partial}_s \Gamma^{*b}_{tl}) - H_t(\dot{\partial}_s \Gamma^{*t}_{kl}) - H^b_{kt}(\dot{\partial}_s \Gamma^{*t}_{bl}) - H_{tk} P^t_{rl}\}]P^r_{jn} - (\dot{\partial}_j a_{lmn}) H_k = 0.$$

Thus, we conclude

Theorem 2.7. *In $WTR - F_n$, H −Ricci tensor H_{jk} is trirecurrent if and only if (2.22) holds.*

Remark 2.1. If the $WTR - F_n$ is affinely connected space, then the new space will be *called WTR − affinely connected space.*

Let us consider WTR − affinely connected space. In view of (1.3c), (1.11b), (1.12) and if $\dot{\partial}_j a_{lmn} = 0$, Eq. (2.15) becomes

(2.23) $H^i_{jkh|l|m|n} = a_{lmn} H^i_{jkh}$.

In view of (1.3c), (1.11b), (1.12) and if $\dot{\partial}_j a_{lmn} = 0$, Eq. (2.18) becomes

(2.24) $H_{jtkh|l|m|n} = a_{lmn} H_{jtkh}$.

In view of (1.3c), (1.11b), (1.12) and if $\dot{\partial}_j a_{lmn} = 0$, Eq. (2.21) becomes

(2,25) $H_{jk|l|m|n} = a_{lmn} H_{jk}$.

Thus, we conclude

Theorem 2.8. *In WTR − affinely connected space, if the directional derivative of covariant tensor field vanish, then the curvature tensor H^i_{jkh}, associate tensor H_{jtkh} and H −Ricci tensor H_{jk} are trirecurrent.*

III. CONCLUSION

The necessary and sufficient condition for some tensors that be trirecurrent has been discussed in W−trirecurrent Finsler space. Also, the relations between the curvature tensors W^i_{jkh}, N^i_{jkh} and H^i_{jkh} have been discussed.

REFERENCES

[1]. A. Abdallah, Study on the relationship between two curvature tensors in Finsler spaces, Journal of Mathematical Analysis and Modeling, 4(2), 112-120, (2023).

[2]. A. Abdallah, A. A. Navlekar and K. P. Ghadle, Special types of generalized $\mathcal{BP}$ −recurrent spaces, Journal of Computer and Mathematical Sciences, 10(5), 972-979, (2019).

[3]. A. Abdallah, A. A. Navlekar, K. P. Ghadle and B. Hardan, Fundamentals and recent studies of Finsler geometry, International Journal of Advances in Applied Mathematics and Mechanics, 10(2), 27-38, (2022).

[4]. F. A. Ahmed and A. A. Abdallah, The conditions for various tensors to be generalized $\mathcal{B}$ −trirecurrent tensor, International Journal of Advanced Research in Science, Communication and Technology, 4(2), 511-517, (2024).

[5]. É. Cartan, Les espaces de Finsler, Actualités, Paris, 79 (1934). 2nd edit. (1971).

[6]. W. H. Hadi, A. M. Hanbala and A. A. Saleem, On projective motion in an N −trirecurrent Finsler space, 7(1), 30 – 44, (2024).

[7]. W. H. Hadi and A. A. Saleem, On pseudo T −trirecurrent Finsler space in Berwald sense, J. E. F., 17(2), 537 – 543, (2023).

[8]. Misra, B. S. Misra, K. Srivastava and R. B. Srivastava, Higher order recurent Finsler spaces with Berwalds curvature tensor field, J. Chem. Bio. Sci. Sei. 4(1), 624 – 631, (2014).

[9]. H. D. Pande and K. K. Gupta, Recurrent Finsler space of third order, Acta Ciencia Indica, 9 (3), 177 – 183, (1980).

[10]. P. N. Pandeyand R. Verma, C^h −birecurrent Finsler space, second conference of the International Academy of Physical Sciences, 13-14, (1997).

[11]. F. Y. Qasem,On normal projection trirecurrent Finsler spaces, Univ. of Aden, J. Nat. and Appl. Sc. 13(2), 339 – 355, (2009).

[12]. F. Y. Qasem and A. A. Abdallah,On generalized $\mathcal{BR}$ −recurrent affinely connected space, Imperial journal of interdisciplinary research, 2(10), 2108-2112, (2016).

[13]. F. Y. Qasem and A. A. Abdallah, On certain generalized $\mathcal{B}R$ −recurrent Finsler space, International Journal of Applied Science and Mathematics, 3(3), 111-114, (2016).

[14]. F. Y. Qasem and A. A. Abdallah, On study generalized $\mathcal{B}R$ −recurrent Finsler space, International Journal of Mathematics And its Application, 4(2-B), 113–121, (2016).

[15]. F. Y. Qasem and A. A. Muhib,On R^h- trirecurrent Finsler space, Journal of the faculty of education, Thamar,University, 6(3), 59 -75, (2008).

[16]. H. Rund, *The differential geometry of Finsler spaces*, Springer-verlag, Berlin Göttingen-Heidelberg, (1959); 2nd (in Russian), Nauka, Moscow, (1981).

[17]. A A. Saleem, On U^h −trirecurrent Finsler space, Univ. Aden j. Nat. and Appl. Sc. 28(1), 1 – 8, (2024).

[18]. A. Saleem and A. A. Abdallah, Study on U^h −birecurrent Finsler space, International Journal of Advanced Research in Science, Communication and Technology, 2(3), 28 – 39, (2022).

[19]. A. Saleem and A. A. Abdallah, Certain identities of C^h in Finsler spaces, International Journal of Advanced Research in Science, Communication and Technology, 3(2), 620-622, (2023).

[20]. A. Saleem and A. A. Abdallah, On pseudo T −birecurrent Finsler space, International Journal of Advanced Multidisciplinary Research and Studies, 2(3), 517 – 522, (2023).

[21]. A. Saleem and A. A. Abdallah, The recurrence property for the projective curvature tensor in Finsler space, International Advanced Research Journal in Science, Engineering and Technology, 11(5), 291 – 297, (2024).

[22]. A. Saleem and A. A. Abdallah, On birecurrent Finsler space for the projective curvature tensor, International Journal of Advanced and Applied Research, 11(6), 95 – 99, (2024).