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Recurrent Finsler Structures with Higher-Order Generalizations Defined by Special Curvature Tensors

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Abstract*: This paper introduces a class of Finsler structures, termed hyper-generalized recurrent Finsler structures. These structures are defined by particular curvature tensors in conjunction with Berwald's covariant differentiation. This paper extends the theory of recurrent Finsler structures by introducing a new class of structures defined by specific curvature tensors and Berwald's covariant differentiation. The findings of this research contribute to a deeper understanding of the intricate interplay between curvature and recurrent properties in Finsler geometry*

Keywords: Finsler structures, Berwald covariant derivative, curvature tensor, geometric properties

I. INTRODUCTION

Finsler geometry, a generalization of Riemannian geometry, offers a powerful framework for studying spaces with anisotropic metric properties. In recent years, there has been a growing interest in recurrent Finsler structures, characterized by the parallel propagation of certain curvature tensors along geodesics. The Berwald's covariant differentiation, a fundamental tool in Finsler geometry, plays a crucial role in our investigation.

The Weyl's projective curvature tensor is a geometric object employed to characterize the curvature of a spacetime or more generally, a pseudo-Riemannian manifold. The Weyl's projective curvature tensor also vanishes precisely when the spacetime is locally isometric to flat spacetime. The study of curvature tensors within Finsler spaces assumes paramount importance due to their pivotal role in characterizing the intrinsic curvature of these spaces. These tensors encapsulate information regarding the deviation of geodesics and the parallel transport of vectors. By scrutinizing the expansion identities for curvature tensors, we seek to uncover deeper connections between the various curvature invariants and to acquire a more comprehensive understanding of the curvature properties of Finsler spaces.

The curvature tensors are fundamental objects in differential geometry. some examples include the Riemannian curvature tensorR^i_{jkh}, Weyl's projective curvature tensorW^i_{jkh}, M −projective curvature tensor $\bar{W}^i_{jkh}$, conformal curvature tensor C^i_{jkh}, conharmonic curvature tensor L^i_{jkh}, concircular curvature tensor M^i_{jkh}, and P_1 −curvature tensor. The Riemannian curvature tensor was introduced by Riemann in 1854. The conformal curvature tensor C^i_{jkh} with another significant curvature tensor, finds extensive applications in differential geometry.

The concept of the three-dimensional of Weyl's space with recurrent curvature was studied and explored by [4, 11]. The analysis of generalized curvature tensors relies on the Berwald curvature tensor has been discussed by [9, 10]. Zafar and Musavvir [6, 26] studied on some properties of W −curvature tensor, Chagpar et al. [12] and Pokhariyal [21] introduced P_1 −Curvature tensor and some tensors with their relativistic significance, Ali and Salman [7] studied some properties of M −projective curvature tensor. The relationship between P^h_{ijk} and R^h_{ijk} in Berwald sense studied by [2]. Berwald derivative $(\mathcal{B}_m)$ of any tensor T^i_j, w. r. t. x^m is defined as [3, 13, 14, 17, 23]

(1.1) $\mathcal{B}_m T_j^i = \partial_m T_j^i - (\dot{\partial}_r T_j^i) G_m^r + T_j^r G_{rm}^i - T_r^i G_{jm}^r$.

The vector y^i and metric function F are vanished identically for Berwald's covariant derivative

(1.2) a) $\mathcal{B}_m F = 0$ and b) $\mathcal{B}_m y^i = 0$.

The metric tensor g_{ij} is not equal to zero for Berwald's covariant derivative [23]

(1.3) $\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h \mathcal{B}_h C_{ijk}$.

The quantities g_{ij} and g^{ij} are related by [1]

(1.4) a) $g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 \ , & if \quad i = k \ , \\ 0 \ , & if \quad i \neq k \ . \end{cases}$ and b) $g_{ij}\, y^i = y_j$.

The tensors R_{jkh}^i and H_{jk}^i give the following identities [5]

(1.5) a) $R_{jkh}^i\, y^j = H_{kh}^i$, b) $H_{kh}^i\, y^k = H_h^i$, c) $H_{ki}^i = H_k$ and d) $H_i^i = (n-1)H$.

The covariant derivative of some tensors are given by [8, 15]

(1.6) a) $\mathcal{B}_m R_{ij} = \lambda_m R_{ij}$ and b) $\mathcal{B}_m \delta_i^k = 0$.

Also

(1.7) a) $\mathcal{B}_m \delta_h^k R_{ij} = \lambda_m \delta_h^k\, R_{ij}$, b) $\mathcal{B}_m g_{ij}\, R_h^k = \lambda_m g_{ij}\, R_h^k$,

c) $\mathcal{B}_m R \delta_k^h g_{ij} = \lambda_m R \delta_k^h g_{ij}$ and d) $\mathcal{B}_m R R_{ij} = \lambda_m R R_{ij}$.

A large number of researchers have presented the following identities in their works [16, 19, 24, 25]

(1.8) a) $C_{ijk}\, y^i = 0$, b) $C_{ijk} = \frac{1}{4}(\dot{\partial}_k \dot{\partial}_i \dot{\partial}_j F^2)$, c) $\dot{\partial}_j\, y^j = 1$, d) $y_j y^j = F^2$,

e) $\delta_j^k y^j = y^k$ and f) $\dot{\partial}_k\, y_j = g_{jk}$.

The derivative for Berwald's $(\mathcal{B}_m)$ of the tensors T_{jkh}^i, T_{jk}^i and T_j^i, w. r. t. x^m are defined as

(1.9) a) $\mathcal{B}_m T_{jkh}^i = \lambda_m T_{jkh}^i$, b) $\mathcal{B}_m T_{jk}^i = \lambda_m T_{jk}^i$ and c) $\mathcal{B}_m T_j^i = \lambda_m T_j^i$.

In this paper we investigate some identities between Weyl's projective curvature tensor W_{jkh}^i and some others curvature tensors by using Berwald covariant derivative. We introduce the basic concepts of the curvature tensors and study the relationships between them. Finally, we apply this expansion and identities to get relationships between different curvature tensors and Weyl's projective curvature tensor W_{jkh}^i.

II. PRELIMINARIES

In this section, we discuss the relationship between Weyl's projective curvature tensor and the following curvature tensors:

Riemanniiancurvature tensor R_{jkh}^i

In the mathematical field of differential geometry, the Riemann curvature tensor is the most common way used to express the curvature of Riemannian manifold. It assigns a tensor to each point of a Riemannian manifold (i.e., it is a tensor field). It is a local invariant of Riemannian metrics which measures the failure of the second covariant derivatives to commute. A Riemannian manifold has zero curvature if and only if it is flat, i.e. locally isometric to the Euclidean space. The curvature tensor can also be defined for any pseudo-Riemannian manifold, or indeed any manifold equipped with an affine connection.

The Riemann curvature tensor is a tool used to describe the curvature of $n-$dimensional spaces such as Riemannian manifold in the field of differential geometry. The Riemann curvature tensor plays an important role in the theories of general relativity and gravity as well as the curvature of a spacetime. It is closely related to the Weyl's projective curvature tensor.Weyl's projective curvature tensor in terms of Riemannian curvature tensor R_{jkh}^i is defined as [26]

(2.1) $W_{jkh}^i = R_{jkh}^i + \frac{1}{(n-1)}(\delta_k^i R_{jh} - R_h^i g_{jk})$.

In (V_4, F), we have

(2.2) $R_{jkh}^i = W_{jkh}^i - \frac{1}{3}(\delta_k^i R_{jh} - R_h^i g_{jk})$.

The tensors W_{jkh}^i and W_{jk}^i give the following identities

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(2.3) a) $W^i_{jkh}\, y^j = W^i_{kh}$, b) $W^i_{jk}\, y^j = W^i_k$, c) $W^i_j\, y^j = 0$ and d) $W^i_i = 0$.

Projective curvature tensor $\overline{W}^i_{jkh}$

The $\overline{W}$ −projective curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conharmonic curvature tensor. It has been studied in a variety of conte xts, including Riemannian geometry, Kähler geometry, and cosmology. The properties of an M −projective curvature tensor were proposed by Pokhariyal and Mishra [20] in 1970. This tensor is described as follows

(2.4) $$\overline{W}(X,Y,Z,T) = \bar{R}(X,Y,Z,T) - \frac{1}{2(n-1)}[S(Y,Z)g(X,T) - S(X,Z)g(Y,T) + g(Y,Z)S(X,T) - g(X,T)S(Y,Z)],$$

where $\overline{W}(X,Y,Z,T) = g(W(X,Y)Z,T)$ and $\bar{R}(X,Y,Z,T) = g(R(X,Y)Z,T)$.

R is the Riemann curvature tensor, S is the Ricci tensor, g is the metric tensor, n is the dimension of the manifold. The $\overline{W}$ −projective curvature tensor has a number of interesting properties. For example, it is invariant under conformal transformations. This means that it is the same for two metrics that are conformally equivalent. The $\overline{W}$ −projective curvature tensor also vanishes if and only if the manifold is Ricci-flat.

The $\overline{W}$ −projective curvature tensor has been used to study a variety of geometric problems. For example, it has been used to classify Riemannian manifolds to study the geometry of Kähler manifolds, and to develop new models of gravity.

The local coordinates expression of equation (2.4) as follows

(2.5) $$\overline{W}_{ljkh} = R_{ljkh} - \frac{1}{2(n-1)}\left[R_{jk}g_{lh} - R_{lk}g_{jh} + g_{jk}R_{lh} - g_{lk}R_{jh}\right].$$

Assuming $n = 4$ and using (2.2) in equation (2.5), then contracting with g^{li}, the M −projective curvature tensor is given by

(2.6) $$\overline{W}^i_{jkh} = W^i_{jkh} - \frac{1}{6}\left(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk}R^i_h - g_{jh}R^i_k\right).$$

Conformal curvature tensor C^i_{jkh}

The conformal curvature tensor, also known as the Weyl's conformal curvature tensor, is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl's tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl's tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force.

The conformal curvature tensor C^h_{ijk} expressed as follows [22, 26]

(2.7) $$C^i_{jkh} = R^i_{jkh} - \frac{1}{2}\left(\delta^i_k R_{jh} - \delta^i_h R_{jk} + R^i_k g_{jh} - R^i_h g_{jk}\right) - \frac{1}{6}R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right).$$

Using (2.2) in equation (2.7), we get

(2.8) $$C^i_{jkh} = W^i_{jkh} - \frac{5}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \frac{1}{6}R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right).$$

Conharmonic curvature tensor L^i_{jkh}

The conharmonic curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and conformal curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

For V_4 , the conharmonic curvature tensor L^i_{jkh} defined as [18]

(2.9) $$L^i_{jkh} = R^i_{jkh} - \frac{1}{2}\left(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k\right).$$

Using (2.2) in equation (2.9), we get

(2.10) $$L^i_{jkh} = W^i_{jkh} + \frac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \frac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right).$$

Concircular curvature tensor M^i_{jkh}

The concircular curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the conformal curvature tensor (also known as the Weyl's curvature tensor) and the projective curvature tensor. The concircular curvature tensor vanishes if and only if the manifold is concircularly flat.

For V_4 , the concircular curvature tensor M^i_{jkh}, is defined as [11]

(2.11) $M^i_{jkh} = R^i_{jkh} - \frac{1}{12}R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right)$.

Using (2.2) in equation (2.11),we get

(2.12) $M^i_{jkh} = W^i_{jkh} - \frac{1}{12}R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right) - \frac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right)$.

P_1 − Curvature tensor

The $P1$ −curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the Ricci curvature tensor and the scalar curvature. The $P1$ −curvature tensor vanishes if and only if the manifold is Ricci-flat and has constant scalar curvature. The tensor $P_1(X,Y,Z,T)$ has been defined as [12]

(2.13) $$P_1(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{2(n-1)}[\, g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z) \\ -g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z)] .$$

The P_1 −curvature tensor in the index notation as [12]

(2.14) $P_{1_{ljkh}} = R_{ljkh} + \frac{1}{2(n-1)}\left[g_{jk}R_{lh} - g_{jh}R_{lk} - g_{lk}R_{jh} + g_{lh}R_{jk}\right]$.

This can be written as

(2.15) $P^i_{1_{jkh}} = R^i_{jkh} + \frac{1}{2(n-1)}\left[g_{jk}R^i_h - g_{jh}R^i_k - \delta^i_k R_{jh} + \delta^i_h R_{jk}\right]$.

In (V_4, F), using (2.2) in equation (2.15), we get

(2.16) $P^i_{1_{jkh}} = W^i_{jkh} + \frac{1}{6}\left[\delta^i_h R_{jk} - g_{jh}R^i_k\right] - \frac{1}{3}\left[\delta^i_k R_{jh} - g_{jk}R^i_h\right]$.

III. EXTENSION GENERALIZED RECURRENT FINSLER SPACES FOR VARIOUS CURVATURES TENSORS

The expansion derivative for Berwald of any curvature tensor is closely related to the Riemann curvature tensor and the Berwald curvature tensor. It vanishes if and only if the Finsler manifold is flat. We introduced the generalized by Berwald covariant derivative $\mathcal{B}_m$ for any tensor T^i_{jkh} that was given by [11]

(3.1) $\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$.

We can write (3.1) by the follows form

$$\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \left[W^i_h(0) - W^i_k(0)\right].$$

From (1.8a), the above equation can be written as

(3.2) $\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \left[W^i_h C_{ijk} y^i - W^i_k C_{ijh} y^i\right]$.

Using (1.8b) in (3.2), we get

(3.3) $\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left[W^i_h \dot\partial_k \dot\partial_i \dot\partial_j F^2 y^i - W^i_k \dot\partial_h \dot\partial_j \dot\partial_i F^2 y^i\right]$.

Applying (1.8c) in (3.3), we get

$$\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left[W^i_h \dot\partial_k \dot\partial_j F^2 - W^i_k \dot\partial_h \dot\partial_j F^2\right] .$$

From (1.8d) the above equation can be written as

(3.4) $\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left[W^i_h \dot\partial_k \dot\partial_j y^j y_j - W^i_k \dot\partial_h \dot\partial_j y^j y_j\right]$.

Applying (1.8c) again in (3.4), we get

$$\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\left[W^i_h \dot\partial_k y_j - W^i_k \dot\partial_h y_j\right].$$

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From (1.8f), we have

(3.5) $\mathcal{B}_m T^i_{jkh} = \lambda_m T^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

From the previous steps, we can conclude the following theorem

Theorem 3.1. The expansion of (1.9a) is given by (3.5).

The dimensionality of Berwald derivative for many curvatures tensors operators will be extended in accordance with theorem 3.1. Mathematical identities are equations that are always true, regardless of the values of the variables involved. They can be used to simplify expressions, solve equations, and prove theorems. we investigated the expansion of Berwald covariant derivative for any curvature tensor that was given in (3.5), i.e.

(3.6) $\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

Suppose that (3.6) holds to investigate the following identities. By taking Berwald covariant derivative for (2.2), we have

(3.7) $\mathcal{B}_m R^i_{jkh} = \mathcal{B}_m W^i_{jkh} - \frac{1}{3}\mathcal{B}_m(\delta^i_k R_{jh} - R^i_h g_{jk})$.

From (1.7a), (1.7b), (3.6) and (3.7), we get

(3.8) $\mathcal{B}_m R^i_{jkh} = \lambda_m \left[W^i_{jkh} - \frac{1}{3}(\delta^i_k R_{jh} - R^i_h g_{jk})\right] + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

By using (2.2) in (3.8), we have

(3.9) $\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

From the previous steps, we can conclude the following

Theorem 3.2. The expansion derivative for Berwald of Riemannian curvature tensor R^i_{jkh} in (2.2) satisfies the equation (3.9).

Take Berwald covariant derivative for (2.6), we have

(3.10) $\mathcal{B}_m \overline{W}^i_{jkh} = \mathcal{B}_m W^i_{jkh} - \frac{1}{6}\mathcal{B}_m(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk} R^i_h - g_{jh} R^i_k)$.

From (1.7a), (1.7b), (3.6) and (3.10), we get

$$\mathcal{B}_m \overline{W}^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$$
$$-\frac{1}{6}\lambda_m(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk} R^i_h - g_{jh} R^i_k).$$

Above equation can be written as

(3.11) $$\mathcal{B}_m \overline{W}^i_{jkh} = \lambda_m \left[W^i_{jkh} - \frac{1}{6}(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk} R^i_h - g_{jh} R^i_k)\right] + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}].$$

From (2.6) and (3.11), we have

(3.12) $\mathcal{B}_m \overline{W}^i_{jkh} = \lambda_m \overline{W}^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

So,we can say

Theorem 3.3. The expansion derivative for Berwald of projective curvature tensor $\overline{W}^i_{jkh}$ in (2.6) satisfies the equation (3.12).

Take Berwald covariant derivative for (2.8), we have

(3.13) $$\mathcal{B}_m C^i_{jkh} = \mathcal{B}_m W^i_{jkh} - \frac{5}{6}\mathcal{B}_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - \frac{1}{6}\mathcal{B}_m R(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{2}\mathcal{B}_m(\delta^i_h R_{jk} - R^i_k g_{jh}) .$$

From (1.7a), (1.7b), (1.7d), (3.6) and (3.13), we get

$$\mathcal{B}_m C^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}] + \frac{1}{2}\lambda_m(\delta^i_h R_{jk} - R^i_k g_{jh})$$
$$-\frac{5}{6}\lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - \frac{1}{6}\lambda_m R(\delta^i_h g_{jk} - \delta^i_k g_{jh}) .$$

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Or

(3.14) $\mathcal{B}_m C^i_{jkh} = \lambda_m[W^i_{jkh} - \frac{5}{6}(\delta^i_k R_{jh} - R^i_h g_{jk}) - \frac{1}{6}R(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{2}(\delta^i_h R_{jk} - R^i_k g_{jh})]$

$$+ \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}] \ .$$

By using (2.8) in (3.14), we have

(3.15) $\mathcal{B}_m C^i_{jkh} = \lambda_m C^i_{jkh} + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

In conclusion, we can determine

Theorem 3.4. The expansion derivative for Berwald of conformal curvature tensor C^h_{ijk} in (2.8) satisfies the equation (3.15).

Take Berwald covariant derivative for (2.10), we have

(3.16) $\mathcal{B}_m L^i_{jkh} = \mathcal{B}_m W^i_{jkh} + \frac{1}{6}\mathcal{B}_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - \frac{1}{2}\mathcal{B}_m(\delta^i_h R_{jk} - R^i_k g_{jh})$.

From (1.7a), (1.7b), (3.6) and (3.16), we get

$$\mathcal{B}_m L^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$$

$+\frac{1}{6}\lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk}) - \frac{1}{2}\lambda_m(\delta^i_h R_{jk} - R^i_k g_{jh})$.

Or

(3.17) $\mathcal{B}_m L^i_{jkh} = \lambda_m\left[W^i_{jkh} + \frac{1}{6}(\delta^i_k R_{jh} - R^i_h g_{jk}) - \frac{1}{2}(\delta^i_h R_{jk} - R^i_k g_{jh})\right]$

$$+ \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}].$$

From (2.10) and (3.17), we get

(3.18) $\mathcal{B}_m L^i_{jkh} = \lambda_m L^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

Thus, we get

Theorem 3.5. The expansion derivative for Berwald of Conharmonic curvature tensor L^i_{jkh} in (2.10) satisfies the equation (3.18).

Take Berwald covariant derivative for (2.12), we have

(3.19) $\mathcal{B}_m M^i_{jkh} = \mathcal{B}_m W^i_{jkh} - \frac{1}{12}\mathcal{B}_m R(g_{jk}\delta^i_h - g_{jh}\delta^i_k) - \frac{1}{6}\mathcal{B}_m(\delta^i_k R_{jh} - R^i_h g_{jk})$.

From (1.7a), (1.7b), (1.7d), (3.6) and (3.19), we get

$$\mathcal{B}_m M^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$$

$-\frac{1}{12}\lambda_m R(g_{jk}\delta^i_h - g_{jh}\delta^i_k) - \frac{1}{6}\lambda_m(\delta^i_k R_{jh} - R^i_h g_{jk})$.

Or

(3.20) $\mathcal{B}_m M^i_{jkh} = \lambda_m\left[W^i_{jkh} - \frac{1}{12}R(g_{jk}\delta^i_h - g_{jh}\delta^i_k) - \frac{1}{6}(\delta^i_k R_{jh} - R^i_h g_{jk})\right]$

$$+\mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}] \ .$$

From (2.12) and (3.20), we have

(3.21) $\mathcal{B}_m M^i_{jkh} = \lambda_m M^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$.

In conclusion, we can determine

Theorem 3.6. The expansion derivative for Berwald of concircular curvature tensor M^i_{jkh} in (2.12) satisfies the equation (3.21).

Take Berwald covariant derivative for (2.16), we have

(3.22) $\mathcal{B}_m P^i_{1\,jkh} = \mathcal{B}_m W^i_{jkh} + \frac{1}{6}\mathcal{B}_m[\delta^i_h R_{jk} - g_{jh} R^i_k] - \frac{1}{3}\mathcal{B}_m[\delta^i_k R_{jh} - g_{jk} R^i_h]$.

From (1.7a), (1.7b), (3.6) and (3.22), we get

$$\mathcal{B}_m P^i_{1_{jkh}} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}]$$

$$+\tfrac{1}{6}\lambda_m[\delta^i_h R_{jk} - g_{jh}R^i_k] - \tfrac{1}{3}\lambda_m[\delta^i_k R_{jh} - g_{jk}R^i_h].$$

Or

(3.23) $$\mathcal{B}_m P^i_{1_{jkh}} = \lambda_m\left[W^i_{jkh} + \tfrac{1}{6}[\delta^i_h R_{jk} - g_{jh}R^i_k] - \tfrac{1}{3}[\delta^i_k R_{jh} - g_{jk}R^i_h]\right] + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}].$$

By using (2.16) in (3.23), we have

(3.24) $$\mathcal{B}_m P^i_{1_{jkh}} = \lambda_m P^i_{1_{jkh}} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}[W^i_h g_{jk} - W^i_k g_{jh}].$$

The proof of theorem is completed, we conclude

Theorem 3.7. The expansion derivative for Berwald of $P1$ −curvature tensor $P^i_{1_{jkh}}$ in (2.16) satisfies the equation (3.24).

Transvecting (3.9) by y^j, using (1.2b), (1.5a) and (1.4b), we get

(3.25) $$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4}[W^i_h y_k - W^i_k y_h].$$

Again, transvecting (3.25) by y^k, using (1.2b), (1.5b), (2.3b), (2.3c), (1.8d) and (1.8e), we get

(3.26) $$\mathcal{B}_m H^i_h = \lambda_m H^i_h + \mu_m(\delta^i_h F^2 - y^i\, y_h) + \tfrac{1}{4}W^i_h F^2.$$

Transvecting (3.6) by y^j, using (1.2b), (2.3a) and (1.4b), we get

(3.27) $$\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4}[W^i_h y_k - W^i_k y_h].$$

Again, transvecting (3.27) by y^k, using (1.2b), (2.3b), (2.3c), (1,8d) and (1.8e), we get

(3.28) $$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m(y^i\, y_k - \delta^i_k F^2) + \tfrac{1}{4}W^i_h F^2.$$

Contracting the indices i and h in the equations (3.25) and (3.26), respectively and using (1.4a), (1.4b), (1.8d), (1.8e), (1.5c), (1.5d) and (2.3d), we get

(3.29) $$\mathcal{B}_m H_k = \lambda_m \mathrm{H}_k + \mu_m(n-1)y_k - \tfrac{1}{4}W^i_k y_i$$

and

(3.30) $$\mathcal{B}_m H = \lambda_m H + \mu_m(n-1)F^2.$$

Therefore, we can say

Corollary 3.1. In covariant derivative for Berwald of first order for H^i_{kh} , H^i_h, W^i_{kh} , W^i_h , H_k and H are given by (3.25), (3.26), (3.27), (3.28), (3.29) and (3.30), respectively.

IV. CONCLUSION

The study of hyper-generalized recurrent Finsler structures has opened up new avenues of research in Finsler geometry. We obtained the relationships between the expansion identities for curvature tensors and other geometric invariants. We give the relationships between Weyl's projective curvature tensor and some others curvature tensors.Further, our findings have the potential to inspire further investigations into the geometric properties of these structures and their connections to other areas of mathematics and physics. By providing a solid foundation for future research, this work aims to stimulate further advancements in the field. The decomposition we have introduced offers a new perspective on the geometric significance of curvature tensors and opens up new avenues for further research.

REFERENCES

[1]. A. Abdallah, Study on the relationship between two curvature tensors in Finsler spaces, Journal of Mathematical Analysis and Modeling, 4(2), (2023), 112-120.

[2]. A. Abdallah and B. Hardan, Two results to clarify the relationship between P^h_{ijk} and R^h_{ijk} with two conections of third order in Finsler spaces, The Scholar Journal for Sciences and Technology, 2(4), (2024), 52 – 59.

[3]. A. Abdallah, K. P. Ghadle, A. A. Navlekar and B. Hardan, Fundamentals and recent studies of Finsler geometry, International Journal of Advances in Applied Mathematics and Mechanics, 10(2), (2022), 27-38.

[4]. H. Abu-Donia, S. Shenawy and A. Abdehameed, The W^* −curvature tensor on relativistic space-times. Kyungpook Mathematical Journal, 60, (2020), 185-195.

[5]. F. A. Ahmed and A. A. Abdallah, The conditions for various tensors to be generalized $\mathcal{B}$ −trirecurrent tensor, International Journal of Advanced Research in Science, Communication and Technology, 4(2) (2024), 511-517.

[6]. Z. Ahsan and M. Ali, On some properties of W −curvature tensor, Palestine Journal of Maths, 3(1), (2014), 61-69.

[7]. M. Ali, M. Salman, N. Pundeer and Y. Jin Suh, On some properties of M −projective curvature tensor in spacetime of general relativity (2022), Accepted.

[8]. A M. AL-Qashbari,Decomposition of curvature tensor filed R^i_{jkh} recurrent spaces of first and second order, Journal of Natural and Applied Sciences, 33(2), (2023), 121-137.

[9]. M. AL-Qashbari, Recurrence decompositions in Finsler space. Journal of Mathematical Analysis and Modelling, 1, (2020), 77-86.

[10]. M. Al-Qashbari, A. A. Abdallah and F. A. Ahmed, On generalized birecurrent Finsler space of mixed covariant derivatives in Cartan sense, International Journal of Research Publication and Reviews, 5(8), (2024), 2834-2840.

[11]. M. AL-Qashbari and A. A. AL-Maisary, Some identities between Wely's cuvature tensor and conformal cuvature tensor. Journal of Applied Sciences and Humanites. Univ. Of Lahej, 1, (2024), 10-23.

[12]. F. Z. Chagpar, G. P. Pokhariyal and S. K. Moindi,P_1 −Curvature tensor in the space time of general relativity. I. Jour. App. Math., 6(1), (2021), 148-152.

[13]. K. P. Ghadle, A. A. Navlekar and A. A. Abdallah, Analysis for Cartan's second curvature tensor in Finsler space, International Journal of Advanced Research in Science, Communication and Technology, 2(3), (2022), 1-5.

[14]. K. P. Ghadle, A. A. Navlekar and A. A. Abdallah, On $\mathcal{B}$ −covariant derivative of first order for some tensors in different spaces, Journal of Mathematical Analysis and Modeling, 2(2), (2021), 30-37.

[15]. K. P. Ghadle, A. A. Navlekar, A. A. Abdallah and B. Hardan, Some tensors in generalized $\mathcal{B}R$ −recurrent Finsler Space. World Journal of Engineering Research and Technology, 9(6), (2023), 42-48.

[16]. A A. Hamoud, A. A. Navlekar, A. A. Abdallah and K. P. Ghadle, On special spaces of $h(hv)$ −torsion tensor C_{jkh} in generalized recurrent Finsler space, Bull. Pure Appl. Sci. Sect. E Math. Stat. 41E (1), (2022), 74-80.

[17]. A. Hamoud, A. A. Navlekar, K. P. Ghadle and A. A. Abdallah, Decomposition for Cartan's second curvature tensor of different order in Finsler spaces, Nonlinear Functional Analysis and Applications, 27(2), (2022), 433-448.

[18]. Y. Ishii, On conharmonic transformation, Tensor (N.S.), 7(2), (1957), 73-80.

[19]. M. Matsumoto, Finsler spaces with the hv −curvature tensors P_{hijk} of a special form, Rep. Math. Phy., 14, (1978), 1-13.

[20]. G. P. Pokhariyal and R. S. Mishra, Curvature tensors and their relativistic significance, Yokohama. J., 19(2), (1971), 97-103.

[21]. G. P. Pokhariyal, Tensors and their relativistic significance (III), Yokohama Mathematical Journal, 21(2), (1973), 115-119.

[22]. M. K. Roopa and S. K. Narasimhamurthy, Conformally transformed Einstein generalized m −th root with curvature properties, Journal of the Tensor Society, 11, (2017), 1-12.

[23]. H. Rund, The differential geometry of Finsler spaces. Springer-Verlag, Berlin Gottingen-Heidelberg, 2nd Edit. (in Russian), Nauka, (Moscow), (1981).

[24]. A A. Saleem and A. A. Abdallah, Certain identities of C^h in Finsler spaces, International Journal of Advanced Research in Science, Communication and Technology, 3(2), (2023), 620-622.

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[25]. K. Yano, The theory of Lie-derivatives and its applications, North-Holland publishing Co. Amsterdam (1957).

[26]. A Zafar and A. Musavvir, On some properties of $W-$curvature tensor, Department of Math., Aligarh Muslim Univ., Aligarh-202002 (India), (2013).