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A Search on Integral Solutions to the Non-Homogeneous Ternary Cubic Equation

$$a x^2 + b y^2 = (a+b) z^3 , a,b > 0$$

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Abstract: *Patterns of non-zero integer solutions to the non-homogeneous ternary cubic equation* $a x^2 + b y^2 = (a+b) z^3 , a,b > 0$. *Some fascinating relations between the solutions are presented*

Keywords: Non-homogeneous cubic, Ternary cubic, Integer solutions

I. INTRODUCTION

It is quite obvious that cubic Diophantine equations are plenty . In this context ,one may refer [1-16] .While analyzing problems on third degree Diophantine equations with three unknowns ,the article presented in [17] came to our reference. The authors of [17] presented only two patterns of solutions in integers. Albeit tacitly , there are other choices of fascinating integer solutions to the cubic equation with three unknowns considered in [17]. The main thrust of this paper is to obtain the new choices of integer solutions. A few relations between the solutions are presented.

II. METHODOLOGY

The non-homogeneous ternary cubic equation under consideration is

$$a x^2 + b y^2 = (a+b) z^3 \tag{1}$$

To start with ,it is observed by scrutiny that (1) is satisfied by

$$x = (m \mp b n) z,$$
$$y = (m \pm a n) z,$$
$$z = m^2 + a b n^2$$

and

$$x = (m \mp b n)\,(m^2 + a b n^2)\,(a+b)^3 ,$$
$$y = (m \pm a n)\,(m^2 + a b n^2)\,(a+b)^3 ,$$
$$z = (m^2 + a b n^2)\,(a+b)^2 .$$

However , there are many more choices of integer solutions to (1).
The process of obtaining other choices of integer solutions to (1) are as below:

Choice 1

The option

$$x = k y \tag{2}$$

in (1) gives

$$(a k^2 + b) y^2 = (a+b) z^3$$

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which is satisfied by

$$y = (a k^2 + b)(a + b)^2 \alpha^{3s}, \qquad (3)$$
$$z = (a k^2 + b)(a + b)\alpha^{2s}, \alpha > 1, s \geq 0$$

From (2) ,one has

$$x = k(a k^2 + b)(a + b)^2 \alpha^{3s} \qquad (4)$$

Thus,(3) and (4) satisfy (1).

Choice 2

The option

$$y = k x \qquad (5)$$

in (1) gives

$$(a + b k^2) x^2 = (a + b) z^3$$

which is satisfied by

$$x = (a + b k^2)(a + b)^2 \alpha^{3s} \qquad (6)$$
$$z = (a + b k^2)(a + b)\alpha^{2s}$$

From (5) ,one has

$$y = k(a + b k^2)(a + b)^2 \alpha^{3s} \qquad (7)$$

Thus,(6) and (7) satisfy (1).

Choice 3

The substitution

$$x = z - bT, y = z + aT \qquad (8)$$

in (1) leads to

$$abT^2 = z^2 (z - 1)$$

which is satisfied by

$$z = 1 + abs^2, \qquad (9)$$
$$T = s(1 + abs^2).$$

From (8) ,one has

$$x = (1 + abs^2)(1 - bs), \qquad (10)$$
$$y = (1 + abs^2)(1 + a s).$$

Thus, the values of x ,y, z given by (9) and (10) satisfy (1).

Observations

i. $ax + by = (a + b)z$

ii. $asx - y + z^2 = 0$

iii. $bsy + x - z^2 = 0$

Note 1

In addition to (8) , one may also consider the substitution

$$x = z + bT, y = z - aT$$

For this option , the corresponding integer solutions to (1) are given by

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$$x = (1 + a b s^2)(1 + b s),$$
$$y = (1 + a b s^2)(1 - a s),$$
$$z = (1 + a b s^2).$$

Choice 4

The substitution of the transformations

$$x = X - b z, y = X + a z \quad (11)$$

in (1) leads to the non-homogeneous quintic equation

$$X^2 = z^2 (z - a b) \quad (12)$$

After performing some algebra , it is seen that the values of z ,X satisfying (12) are given by

$$z = z_n = a b + (s + n)^2, \quad X = X_n = (a b + (s + n)^2)(s + n) \quad (13)$$

From (11) ,it is obtained that

$$x_n = (a b + (s + n)^2)(s + n - b), \quad y_n = (a b + (s + n)^2)(s + n + a). \quad (14)$$

Thus, the values of x ,y, z given by (13) and (14) satisfy (1).

Observations

i. $y_n - x_n = (a + b) z_n$

ii. $(b y_n + a x_n)^2 = (z_n - a b)(y_n - x_n)^2$

iii. $(x_n + b z_n)^2 = (y_n - a z_n)^2 = z_n^2 (z_n - a b)$

iv. $(y_n - x_n)^2 [(x_n + b z_n)^2] = (y_n - x_n)^2 [(y_n - a z_n)^2] = (b y_n + a x_n)^2 z_n^2$

Note 2

In addition to (11) , one may also consider the substitution

$$x = X + b z, y = X - a z$$

For this option , the corresponding integer solutions to (1) are given by

$$x_n = (a b + (s + n)^2)(s + n + b),$$
$$y_n = (a b + (s + n)^2)(s + n - a),$$
$$z_n = (a b + (s + n)^2)$$

III. CONCLUSION

In this paper , we have presented integer solutions to the cubic equation in title which are different from the solutions given in [17]. As the cubic equations are plenty, one may search for other forms to cubic equations to determine their solutions in integers utilizing substitution technique and factorization method.

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