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# The Conditions for Various Tensors to be Generalized β-Trirecurrent Tensor

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**Abstract:** In this paper, we conclude the conditions for these tensors $H^i_{jkh}$,$K^i_{jkh}$, $R^i_{jkh}$, $H^i_{kh}$, $(H_{hk} - H_{kh})$, $H^i_j$, $K_{jk}$, $K_j$, $R_{jk}$ and $R_j$ to be generalized $\beta -$ trirecurrent

**Keywords:** Generalized $\beta H -$trirecurrent Finsler space, Berwald curvature tensor $H^i_{jkh}$, $\beta -$covariant derivative

## I. INTRODUCTION

The necessary and sufficient conditions for some curvature tensors that satisfy the generalized recurrent and birecurrent in sense of Berwald have been studied by [3, 5, 6, 7, 8, 9, 11, 14, 15, 16, 17, 19]. Recently, some conditions for $R^i_{jkh}$, $P^i_{jkh}$ and $H^i_{jkh}$ that satisfy the generalized trire current in sense of Berwald have been discussed by [4, 12, 18].

An$n$–dimensional Finsler space $F_n$ equipped with the metric function $F(x,y)$ satisfying the request conditions [10, 20, 22], we have

(1.1) a) $\delta^i_j y_i = y_j$, b) $\delta^i_j y^j = y^i$, c) $\delta^i_j g_{ir} = g_{jr}$, d) $\delta^i_j g^{jk} = g^{ik}$, e) $y^k y_k = F^2$,

f) $y_i = g_{ij} y^j$, g) $\dot{\partial}_j y_h = g_{jh}$ and h) $g_{ij} g^{ik} = \delta^k_j = \begin{cases} 1 & if \ j = k, \\ 0 & if \ j \neq k. \end{cases}$

The $(h)hv -$torsion tensor which is positively homogeneous of degree $-1$ in $y^i$ and symmetric in all its indices introduced and defined by [2, 13, 21]

$C_{ijk} = \frac{1}{2} \dot{\partial}_i g_{jk} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2$.

And satisfies

(1.2) a) $C_{ijk} y^i = C_{kij} y^i = C_{jki} y^i = 0$, b) $C^h_{jk} g_{ih} = C_{ijk}$ and c) $\delta^i_j C_{kil} = C_{kjl}$.

Berwald's covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field $T^i_j$ with respect to $x^k$ is given by [20]

$$\beta_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

The connection parameter $G^i_{jk}$ of Berwald is connected with Cartan's connection parameter $\Gamma^{*i}_{jk}$ by

$$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|h} y^h.$$

In view of Euler's theorem, we have

(1.3) $G^i_{jkh} y^j = G^i_{hjk} y^j = G^i_{khj} y^j = 0$.

Berwald's covariant derivative of $y^i$ vanish identically, i.e.

(1.4) $\beta_k y^i = 0$.

Berwald's covariant differential with respect to $x^h$ and the partial differentiation with respect to $y^k$ commute according to [10]

(1.5) $(\dot{\partial}_k \beta_h - \beta_h \dot{\partial}_k) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$

for an arbitrary tensor field $T^i_j$. The curvature tensor $H^i_{jkh}$, $h(hv)-$ torsion tensor $H^i_{kh}$, deviation tensor $H^i_h$ and curvature vector $H_k$ satisfy the following

(1.6) a) $\dot{\partial}_j H^i_{kh} = H^i_{jkh}$, b) $H^i_{kh} y^k = -H^i_{hk} y^k = H^i_h$, c) $H^i_k y^k = 0$,

d) $H^i_{jkh} y^k = H^i_{jh}$, e) $H^i_i = (n-1) H$ and f) $H_{hk} - H_{kh} = H^i_{ikh}$

The curvature tensor $K^{\,i}_{jkh}$, $K$ – Ricci tensor $K_{jk}$ and curvature vector $K_j$ are given by [10]

(1.7) a) $K^i_{jki} = K_{jk}$ , b) $K^i_{jkh} = R^i_{jkh} - C^i_{js}\, H^s_{kh}$ and c) $K_j = K_{jk}y^k$ .

Berwald curvature tensor $H^i_{jkh}$ and Cartan's fourth curvature tensor $K^i_{jkh}$ are connected by

(1.8) $H^i_{jkh} = K^i_{jkh} + y^s\,(\dot{\partial}_j K^i_{skh})$ .

The $R$ – Ricci tensor $R_{jk}$ satisfy the following [1, 20]

(1.9) a) $R^{\,r}_{jkr} = R_{jk}$ and b) $R_{jk}\, y^k = R_j$ .

The generalized $\beta H$ –trirecurrent Finsler space introduced by Qasem and Ahmed which characterized by [18]

(1.10) $\beta_\ell\beta_m\beta_n H^i_{jkh} = c_{\ell mn}H^i_{jkh} + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - 2y^r b_{mn}\beta_r(\delta^i_k C_{jh\ell} - \delta^i_h C_{jk\ell})$
$-2y^r w_{\ell n}\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) - 2y^r \mu_n\beta_\ell\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}),\quad H^i_{jkh} \neq 0$ .

This space denoted it by $G\beta H - TRF_n$ . And the tensor will be called a generalized $\mathcal{B}$ –trirecurrent tensor.

The **Necessary and Sufficient Condition for Some Tensors to be Generalized βH − Trirecurrent**

Let us consider $G\beta H - TRF_n$. Differentiating (1.11) partially with respect to $y^j$, using (1.6a) and (1.1g), we get

$$\dot{\partial}_j(\beta_\ell\beta_m\beta_n H^i_{kh}) = (\dot{\partial}_j\, c_{\ell mn})\, H^i_{kh} + c_{\ell mn}H^i_{jkh} + (\dot{\partial}_j\, d_{\ell mn})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using the commutative formula (1.5) for $(\beta_m\beta_n H^i_{kh})$ in above equation, we get

$$\beta_\ell\dot{\partial}_j(\beta_m\beta_n H^i_{kh}) - (\beta_r\beta_n H^i_{kh})G^r_{j\ell m} - (\beta_m\beta_r H^i_{kh})G^r_{j\ell n} + (\beta_m\beta_n H^r_{kh})G^i_{j\ell r} - (\beta_m\beta_n H^i_{rh})G^r_{j\ell k}$$
$$-(\beta_m\beta_n H^i_{kr})G^r_{j\ell h} = (\dot{\partial}_j\, c_{\ell mn})\, H^i_{kh} + c_{\ell mn}H^i_{jkh} + (\dot{\partial}_j\, d_{\ell mn})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Again, applying the commutative formula (1.5) for $(\beta_n H^i_{kh})$ in above equation, we get

$$\beta_\ell\{\beta_m\dot{\partial}_j(\beta_n H^i_{kh}) - (\beta_r H^i_{kh})G^r_{jmn} + (\beta_n H^r_{kh})G^i_{jmr} - (\beta_n H^i_{rh})G^r_{jmk}(\beta_n H^i_{kr})G^r_{jmh}\}$$
$$-(\beta_r\beta_n H^i_{kh})G^r_{j\ell m} - (\beta_m\beta_r H^i_{kh})G^r_{j\ell n} + (\beta_m\beta_n H^r_{kh})G^i_{j\ell r} - (\beta_m\beta_n H^i_{rh})G^r_{j\ell k} - (\beta_m\beta_n H^i_{kr})G^r_{j\ell h}$$
$$= (\dot{\partial}_j\, c_{\ell mn})\, H^i_{kh} + c_{\ell mn}H^i_{jkh} + (\dot{\partial}_j\, d_{\ell mn})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Again, applying the commutative formula (1.5) for $(H^i_{kh})$ in above equation, we get

(2.1)
$$\beta_\ell\,\beta_m\beta_n H^i_{jkh} + (\beta_\ell\beta_m H^r_{kh})G^i_{jnr} + (\beta_\ell H^r_{kh})(\beta_m G^i_{jnr}) + (\beta_m H^r_{kh})(\beta_\ell G^i_{jnr})$$
$$+ H^r_{kh}(\beta_\ell\beta_m G^i_{jnr}) - (\beta_\ell\beta_m H^i_{rh})G^r_{jnk} - (\beta_\ell H^i_{rh})(\beta_m G^r_{jnk}) - (\beta_m H^i_{rh})(\beta_\ell G^r_{jnk})$$
$$- H^i_{rh}(\beta_\ell\beta_m G^r_{jnk}) - (\beta_\ell\beta_m H^i_{kr})G^r_{jnh} - (\beta_\ell H^i_{kr})(\beta_m G^r_{jnh}) - (\beta_m H^i_{kr})(\beta_\ell G^r_{jnh})$$
$$-H^i_{kr}(\beta_\ell\beta_m G^r_{jnh}) - (\beta_\ell\beta_r H^i_{kh})G^r_{jmn} - (\beta_r H^i_{kh})(\beta_\ell G^r_{jmn}) + (\beta_\ell\beta_n H^r_{kh})G^i_{jmr}$$
$$+ (\beta_n H^r_{kh})(\beta_\ell G^i_{jmr}) + (\beta_\ell\beta_n H^i_{rh})G^r_{jmk} - (\beta_n H^i_{rh})(\beta_\ell G^r_{jmk}) - (\beta_\ell\beta_n H^i_{kr})G^r_{jmh}$$
$$- (\beta_n H^i_{kr})(\beta_\ell G^r_{jmh}) - (\beta_r\,\beta_n H^i_{kh})G^r_{j\ell m} - (\beta_m\,\beta_r H^i_{kh})G^r_{j\ell n} + (\beta_m\beta_n H^r_{kh})G^i_{j\ell r}$$
$$- (\beta_m\beta_n H^i_{rh})G^r_{j\ell k} - (\beta_m\beta_n H^i_{kr})G^r_{j\ell h} = (\dot{\partial}_j c_{\ell mn})\, H^i_{kh} + c_{\ell mn}H^i_{jkh}$$
$$+(\dot{\partial}_j d_{\ell mn})(\delta^i_k y_h - \delta^i_h y_k) + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

This shows that

(2.2) $\beta_\ell\,\beta_m\beta_n H^i_{jkh} = c_{\ell mn}H^i_{jkh} + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$

if and only if

(2.3)
$$(\beta_\ell\beta_m H^r_{kh})G^i_{jnr} + (\beta_\ell H^r_{kh})(\beta_m G^i_{jnr}) + (\beta_m H^r_{kh})(\beta_\ell G^i_{jnr}) + H^r_{kh}(\beta_\ell\beta_m G^i_{jnr})$$
$$- (\beta_\ell\beta_m H^i_{rh})G^r_{jnk} - (\beta_\ell H^i_{rh})(\beta_m G^r_{jnk}) + (\beta_m H^i_{rh})(\beta_\ell G^r_{jnk}) - H^i_{rh}(\beta_\ell\beta_m G^r_{jnk})$$
$$-H^i_{rh}(\beta_\ell\beta_m G^r_{jnk}) - (\beta_\ell\beta_m H^i_{kr})G^r_{jnh} - (\beta_\ell H^i_{kr})(\beta_m G^r_{jnh}) + (\beta_m H^i_{kr})(\beta_\ell G^r_{jnh})$$
$$-H^i_{kr}(\beta_\ell\beta_m G^r_{jnh}) - (\beta_\ell\beta_r H^i_{kh})G^r_{jmn} - (\beta_r H^i_{kh})(\beta_\ell G^r_{jmn}) + (\beta_\ell\beta_n H^r_{kh})G^i_{jmr}$$
$$- (\beta_n H^r_{kh})(\beta_\ell G^i_{jmr}) + (\beta_\ell\beta_n H^i_{rh})G^r_{jmk} + (\beta_n H^i_{rh})(\beta_\ell G^r_{jmk}) - (\beta_\ell\beta_n H^i_{kr})G^r_{jmh}$$
$$- (\beta_n H^i_{kr})(\beta_\ell G^r_{jmh}) - (\beta_r\,\beta_n H^i_{kh})G^r_{j\ell m} - (\beta_m\,\beta_r H^i_{kh})G^r_{j\ell n} + (\beta_m\beta_n H^r_{kh})G^i_{j\ell r}$$
$$- (\beta_m\beta_n H^i_{rh})G^r_{j\ell k} - (\beta_m\beta_n H^i_{kr})G^r_{j\ell h} = (\dot{\partial}_j\, c_{\ell mn})\, H^i_{kh} + (\dot{\partial}_j\, d_{\ell mn})(\delta^i_k y_h - \delta^i_h y_k).$$

Thus, we conclude

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**Theorem 2.1.** *In $G\beta H - TRF_n$, Berwald's covariant derivative of third order for the curvature tensor $H^i_{jkh}$ is given by (2.2) if and only if (2.3) holds.*

Transvecting (2.1) by $y^k$, using (1.4), (1.6d), (1.1b), (1.1f), (1.6b) and (1.3), we get

(2.4) $$\beta_\ell \beta_m\beta_n H^i_{jh} + (\beta_\ell \beta_m H^r_h)G^i_{jnr} + (\beta_\ell H^r_h)(\beta_m G^i_{jnr}) + (\beta_m H^r_h)(\beta_\ell G^i_{jnr}) + H^r_h(\beta_\ell \beta_m G^i_{jnr}) - (\beta_\ell \beta_m H^i_r)G^r_{jnh} - (\beta_\ell H^i_r)(\beta_m G^r_{jnh}) - (\beta_m H^i_r)(\beta_\ell G^r_{jnh}) - H^i_r(\beta_\ell \beta_m G^r_{jnh}) - (\beta_\ell \beta_r H^i_h)G^r_{jmn} - (\beta_r H^i_h)(\beta_\ell G^r_{jmn}) + (\beta_\ell \beta_n H^r_h)G^i_{jmr} + (\beta_n H^r_h)(\beta_\ell G^i_{jmr}) - (\beta_\ell \beta_n H^i_r) G^r_{jmh} - (\beta_n H^i_r)(\beta_\ell G^r_{jmh}) - (\beta_r \beta_n H^i_h)G^r_{j\ell m} - (\beta_m \beta_r H^i_h)G^r_{j\ell n} + (\beta_m\beta_n H^r_h)G^i_{j\ell r} - (\beta_m \beta_n H^i_r)G^r_{j\ell h} = (\dot\partial_j c_{\ell mn})H^i_h + c_{\ell mn}H^i_{jh} + (\dot\partial_j d_{\ell mn})(y^i y_h - \delta^i_h F^2) + d_{\ell mn}(y^i g_{jh} - \delta^i_h y_j).$$

This shows that

(2.5) $$\beta_\ell \beta_m\beta_n H^i_{jh} = c_{\ell mn}H^i_{jh} + d_{\ell mn}(y^i g_{jh} - \delta^i_h y_j)$$

if and only if

(2.6) $$(\beta_\ell \beta_m H^r_h)G^i_{jnr} + (\beta_\ell H^r_h)(\beta_m G^i_{jnr}) + (\beta_m H^r_h)(\beta_\ell G^i_{jnr}) + H^r_h(\beta_\ell \beta_m G^i_{jnr})(\beta_\ell \beta_m H^i_r)G^r_{jnh} - (\beta_\ell H^i_r)(\beta_m G^r_{jnh}) - (\beta_m H^i_r)(\beta_\ell G^r_{jnh}) - H^i_r(\beta_\ell \beta_m G^r_{jnh}) - (\beta_\ell \beta_r H^i_h)G^r_{jmn} - (\beta_r H^i_h)(\beta_\ell G^r_{jmn}) + (\beta_\ell \beta_n H^r_h)G^i_{jmr} + (\beta_n H^r_h)(\beta_\ell G^i_{jmr}) - (\beta_\ell\beta_n H^i_r)G^r_{jmh} - (\beta_n H^i_r)(\beta_\ell G^r_{jmh}) - (\beta_r \beta_n H^i_h)G^r_{j\ell m} - (\beta_m \beta_r H^i_h)G^r_{j\ell n} + (\beta_m\beta_n H^r_h)G^i_{j\ell r} - (\beta_m \beta_n H^i_r)G^r_{j\ell h} + (\dot\partial_j c_{\ell mn})H^i_h + (\dot\partial_j d_{\ell mn})(y^i y_h - \delta^i_h F^2) = 0.$$

Thus, we conclude

**Theorem 2.2.** *In $G\beta H - TRF_n$, Berwald's covariant derivative of third order for the $h(v) -$ torsion tensor $H^i_{jh}$ is given by (2.5) if and only if (2.6) holds.*

Transvecting (2.4) by $y^h$, using (1.4), (1.6b), (1.6c), (1.3), (1.1b), (1.1e) and (1.1f), we get

$\beta_\ell \beta_m\beta_n H^i_j = c_{\ell mn}H^i_j$ .

Contracting the indies $i$ and $j$ in above equation and using (1.6e), we get

$\beta_\ell \beta_m\beta_n H = c_{\ell mn} H$ .

Thus, we conclude

**Corollary 2.1.** *In $G\beta H - TRF_n$, the deviation tensor $H^i_j$ and scalar curvature $H$ behave as trirecurrent.*

Contracting the indices $i$ and $j$ in (2.1), using (1.6f), (1.1c) and the symmetric property of metric tensor $g_{ij}$, we get

$$\beta_\ell \beta_m\beta_n H_{kh} + (\beta_\ell\beta_m H^r_{kh})G^p_{pnr} + (\beta_\ell H^r_{kh})(\beta_m G^p_{pnr}) + (\beta_m H^r_{kh})(\beta_\ell G^p_{pnr}) + H^r_{kh}(\beta_\ell\beta_m G^p_{pnr}) - (\beta_\ell\beta_m H^p_{rh})G^r_{pnk} - (\beta_\ell H^p_{rh})(\beta_m G^r_{pnk}) + (\beta_m H^p_{rh})(\beta_\ell G^r_{pnk}) - H^p_{rh}(\beta_\ell\beta_m G^r_{pnk}) - (\beta_\ell\beta_m H^p_{kr})G^r_{pnh} - (\beta_\ell H^p_{kr})(\beta_m G^r_{pnh}) + (\beta_m H^p_{kr})(\beta_\ell G^r_{pnh}) - H^p_{kr}(\beta_\ell\beta_m G^r_{pnh}) - (\beta_\ell\beta_r H^p_{kr})G^r_{pmn} - (\beta_r H^p_{kr})(\beta_\ell G^r_{pmn}) + (\beta_\ell\beta_n H^r_{kh})G^p_{pmr} - (\beta_n H^r_{kh})(\beta_\ell G^p_{pmr}) + (\beta_\ell\beta_n H^p_{rh})G^r_{pmk} + (\beta_n H^p_{rh})(\beta_\ell G^r_{pmk}) - (\beta_\ell\beta_n H^p_{kr})G^r_{pmh} - (\beta_n H^p_{kr})(\beta_\ell G^r_{pmh}) - (\beta_r \beta_n H^p_{kh})G^r_{p\ell m} - (\beta_m \beta_r H^p_{kh})G^r_{p\ell m} + (\beta_m\beta_n H^r_{kh})G^p_{p\ell r} - (\beta_m\beta_n H^p_{rh})G^r_{p\ell k} - (\beta_m\beta_n H^p_{kr})G^r_{p\ell h} = (\dot\partial_p c_{\ell mn}H^p_{kh}) + c_{\ell mn}(H_{kh} - H_{hk}) + (\dot\partial_p d_{\ell mn})(\delta^p_k y_h - \delta^p_h y_k).$$

This shows that

(2.7) $$\beta_\ell \beta_m\beta_n(H_{hk} - H_{kh}) = c_{\ell mn}(H_{hk} - H_{kh})$$

if and only if

(2.8) $$(\beta_\ell\beta_m H^r_{kh})G^p_{pnr} + (\beta_\ell H^r_{kh})(\beta_m G^p_{pnr}) + (\beta_m H^r_{kh})(\beta_\ell G^p_{pnr}) + H^r_{kh}(\beta_\ell\beta_m G^p_{pnr}) - (\beta_\ell\beta_m H^p_{rh})G^r_{pnk} - (\beta_\ell H^p_{rh})(\beta_m G^r_{pnk}) + (\beta_m H^p_{rh})(\beta_\ell G^r_{pnk}) - H^p_{rh}(\beta_\ell\beta_m G^r_{pnk}) - (\beta_\ell\beta_m H^p_{kr})G^r_{pnh} - (\beta_\ell H^p_{kr})(\beta_m G^r_{pnh}) + (\beta_m H^p_{kr})(\beta_\ell G^r_{pnh}) - H^p_{kr}(\beta_\ell\beta_m G^r_{pnh}) - (\beta_\ell\beta_r H^p_{kr})G^r_{pmn} - (\beta_r H^p_{kr})(\beta_\ell G^r_{pmn}) + (\beta_\ell\beta_n H^r_{kh})G^p_{pmr} - (\beta_n H^r_{kh})(\beta_\ell G^p_{pmr})$$

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$$+(\beta_\ell\beta_n H^p_{rh})G^r_{pmk}+(\beta_n H^p_{rh})(\beta_\ell G^r_{pmk})-(\beta_\ell\beta_n H^p_{kr})G^r_{pmh}-(\beta_n H^p_{kr})(\beta_\ell G^r_{pmh})$$
$$-(\beta_r\,\beta_n H^p_{kh})G^r_{p\ell m}-(\beta_m\,\beta_r H^p_{kh})G^r_{p\ell m}+(\beta_m\beta_n H^r_{kh})G^p_{p\ell r}-(\beta_m\beta_n H^p_{rh})G^r_{p\ell k}$$
$$-(\beta_m\beta_n H^p_{kr})G^r_{p\ell h}=(\dot\partial_p\,c_{\ell mn})\,H^p_{kh}+(\dot\partial_p\,d_{\ell mn})(\delta^p_k y_h-\delta^p_h y_k)\,.$$

Thus, we conclude

**Theorem 2.3.** *In $G\beta H-TRF_n$, the tensor $(H_{hk}-H_{kh})$ behaves as trirecurrnet if and only if (2.8) holds.*

Taking the $\beta-$covariant derivative of third order for (1.8) with respect to $x^n$, $x^m$ and $x^\ell$, successively, we get

$\beta_\ell\,\beta_m\beta_n H^i_{jkh}=\beta_\ell\,\beta_m\beta_n K^i_{jkh}+\beta_\ell\,\beta_m\beta_n\{y^s(\dot\partial_j K^i_{skh})\}$.

Using the condition (1.10) in above equation, then using (1.8), we get

(2.9) $c_{\ell mn}K^i_{jkh}+c_{\ell mn}\{y^s(\dot\partial_j K^i_{skh})\}+d_{\ell mn}(\delta^i_k g_{jh}-\delta^i_h g_{jk})-2y^r b_{mn}\beta_r(\delta^i_k C_{jh\ell}-\delta^i_h C_{jk\ell})$
$-2y^r w_{\ell n}\beta_r(\delta^i_k C_{jhm}-\delta^i_h C_{jkm})-2y^r\mu_n\beta_\ell\beta_r(\delta^i_k C_{jhm}-\delta^i_h C_{jkm})$
$=\beta_\ell\,\beta_m\beta_n K^i_{jkh}+\beta_\ell\,\beta_m\beta_n\{y^s(\dot\partial_j K^i_{skh})\}$.

This shows that

(2.10) $\beta_\ell\,\beta_m\beta_n K^i_{jkh}=c_{\ell mn}K^i_{jkh}+d_{\ell mn}(\delta^i_k g_{jh}-\delta^i_h g_{jk})-2y^r b_{mn}\beta_r(\delta^i_k C_{jh\ell}-\delta^i_h C_{jk\ell})$
$-2y^r w_{\ell n}\beta_r(\delta^i_k C_{jhm}-\delta^i_h C_{jkm})-2y^r\mu_n\beta_\ell\beta_r(\delta^i_k C_{jhm}-\delta^i_h C_{jkm})$

if and only if

(2.11) $\beta_\ell\,\beta_m\beta_n\{y^s(\dot\partial_j K^i_{skh})\}=c_{\ell mn}\{y^s(\dot\partial_j K^i_{skh})\}$.

Thus, we conclude

**Theorem 2.4.** *In $G\beta H-TRF_n$, Cartan's fourth curvature tensor $K^i_{jkh}$ is generalized $\mathcal{B}-$trirecurrent if and only if the tensor $\{y^s(\dot\partial_j K^i_{skh})\}$ behaves as trirecurrent*

Contracting the indies $i$ and $h$ in (2.9) and using (1.7a), (1.1c), (1.1h) and (1.2c), we get

(2.12) $c_{\ell mn}K_{jk}+c_{\ell mn}\{y^s(\dot\partial_j K_{sk})\}+d_{\ell mn}(1-n)g_{jk}-2y^r b_{mn}\beta_r(1-n)C_{jk\ell}-2y^r w_{\ell n}\beta_r(1-n)C_{jkm}-2y^r\mu_n\beta_\ell\beta_r(1-n)C_{jkm}=\beta_\ell\,\beta_m\beta_n K_{jk}+\beta_\ell\,\beta_m\beta_n\{y^s(\dot\partial_j K_{sk})\}$.

This shows that

(2.13) $\beta_\ell\,\beta_m\beta_n K_{jk}=c_{\ell mn}K_{jk}+(1-n)\,d_{\ell mn}\,g_{jk}$

if and only if

(2.14) $\beta_\ell\,\beta_m\beta_n\{y^s(\dot\partial_j K_{sk})\}=c_{\ell mn}\{y^s(\dot\partial_j K_{sk})\}+-2(1-n)\,y^r b_{mn}\beta_r C_{jk\ell}$
$-2y^r w_{\ell n}\beta_r(1-n)C_{jkm}-2y^r\mu_n\beta_\ell\beta_r(1-n)C_{jkm}$.

Thus, we conclude

**Theorem 2.5.** *In $\beta H-TRF_n$, the K – Ricci tensor $K_{jk}$ is non-vanishing if and only if (2.14) holds.*

Transvecting (2.12) by $y^k$, using (1.7c), (1.1f) and (1.2a), we get

(2.15) $\beta_\ell\,\beta_m\beta_n K_j=c_{\ell mn}K_j$

if and only if

(2.16) $\beta_\ell\,\beta_m\beta_n\{y^s(\dot\partial_j K_s)\}=c_{\ell mn}\{y^s(\dot\partial_j K_s)\}+d_{\ell mn}(1-n)y_j$.

The equation (2.16) shows that tensor $\{y^s(\dot\partial_j K_s)\}$, can't vanish, because the vanishing of it would implies the vanishing of the covariant vector field $d_{\ell mn}$, i.e. $d_{\ell mn}=0$, contradiction. Thus, we conclude

**Corollary 2.2.** *In $\beta H-TRF_n$, the curvature vector $K_j$ behaves as trirecurrent if and only if the tensor $\{y^s(\dot\partial_j K_s)\}$ is non – vanishing.*

Using (1.8) in (1.7b), we get

(2.17) $R^i_{jkh}=H^i_{jkh}-y^s(\dot\partial_j K^i_{skh})+C^i_{js}H^s_{kh}$.

Taking the $\beta-$covariant derivative of third order for (2.17) with respect to $x^n$, $x^m$ and $x^\ell$, successively, we get

(2.18) $\beta_\ell\, \beta_m \beta_n R^i_{jkh} = \beta_\ell\, \beta_m \beta_n H^i_{jkh} - \beta_\ell\, \beta_m \beta_n \{ y^s (\dot{\partial}_j K^i_{skh}) - C^i_{js} H^s_{kh} \}$ .

Using the condition (1.10) in (2.18) and in view of (2.17), we get

(2.19) $\beta_\ell\, \beta_m \beta_n R^i_{jkh} = c_{\ell mn} R^i_{jkh} + c_{\ell mn} \{ y^s (K^i_{skh}) - C^i_{js} H^s_{kh} \} + d_{\ell mn} (\delta^i_k g_{jh} - \delta^i_h g_{jk})$
$-2y^r b_{mn} \beta_r (\delta^i_k C_{jh\ell} - \delta^i_h C_{jk\ell}) - 2y^r w_{\ell n} \beta_r (\delta^i_k C_{jhm} - \delta^i_h C_{jkm})$
$-2y^r \mu_n \beta_\ell \beta_r (\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) - \beta_\ell\, \beta_m \beta_n \{ y^s (K^i_{skh}) - C^i_{js} H^s_{kh} \}$ .

This shows that

(2.20) $\beta_\ell\, \beta_m \beta_n R^i_{jkh} = c_{\ell mn} R^i_{jkh} + d_{\ell mn} (\delta^i_k g_{jh} - \delta^i_h g_{jk}) - 2y^r b_{mn} \beta_r (\delta^i_k C_{jh\ell} - \delta^i_h C_{jk\ell})$
$-2y^r w_{\ell n} \beta_r (\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) - 2y^r \mu_n \beta_\ell \beta_r (\delta^i_k C_{jhm} - \delta^i_h C_{jkm})$.

if and only if

(2.21) $\beta_\ell\, \beta_m \beta_n \{ y^s (K^i_{skh}) - C^i_{js} H^s_{kh} \} = c_{\ell mn} \{ y^s (K^i_{skh}) - C^i_{js} H^s_{kh} \}$ .

Thus, we conclude

**Theorem 2.6.** *In $\beta H - TRF_n$ , Cartan's third curvature tensor $R^i_{jkh}$ is generalized $\beta$ −trirecurrent if and only if the tensor $\{ y^s (K^i_{skh}) - C^i_{js} H^s_{kh} \}$ behave as trirecurrent.*

Contracting the indies $i$ and $h$ in (2.19), using (1.9a), (1.7a), (1.1c), (1.1h) and (1.2c), we get

(2.22) $\beta_\ell\, \beta_m \beta_n R_{jk} = c_{\ell mn} R_{jk} + c_{\ell mn} \{ y^s (K_{sk}) - C^p_{js} H^s_{pk} \} + d_{\ell mn} (1-n) g_{jk} - 2y^r b_{mn} \beta_r (1-n) C_{jk\ell}$
$-2y^r w_{\ell n} \beta_r (1-n) C_{jkm} - 2y^r \mu_n \beta_\ell \beta_r (1-n) C_{jkm} - \beta_\ell\, \beta_m \beta_n \{ y^s (K_{sk}) - C^p_{js} H^s_{pk} \}$.

This shows that

(2.23) $\beta_\ell\, \beta_m \beta_n R_{jk} = c_{\ell mn} R_{jk}$

if and only if

(2.24) $\beta_\ell\, \beta_m \beta_n \{ y^s (K_{sk}) - C^p_{js} H^s_{pk} \} = c_{\ell mn} \{ y^s (K_{sk}) - C^p_{js} H^s_{pk} \} + d_{\ell mn} (1-n) g_{jk}$
$-2y^r b_{mn} \beta_r (1-n) C_{jk\ell} - 2y^r w_{\ell n} \beta_r (1-n) C_{jkm} - 2y^r \mu_n \beta_\ell \beta_r (1-n) C_{jkm}$ .

Transvecting (2.22) by $y^k$, using (1.9b), (1.7c), (1.1f) and (1.2a), we get

(2.25) $\beta_\ell\, \beta_m \beta_n R_j = c_{\ell mn} R_j$

if and only if

(2.2) $\beta_\ell\, \beta_m \beta_n \{ y^s (K_s) - C^p_{js} H^s_p \} = c_{\ell mn} \{ y^s (K_s) - C^p_{js} H^s_p \} + d_{\ell mn} (1-n) y_j$ .

Thus, we conclude

**Corollary 2.3.** *In $G\beta H - TRF_n$, the R – Ricci tensor $R_{jk}$ and curvature vector $R_j$ behave as trirecurrent if and only if the tensors $\{ y^s (K_{sk}) - C^p_{js} H^s_{pk} \}$ and $\{ y^s (K_s) - C^p_{js} H^s_p \}$ are non-vanishing*

## II. CONCLUSION

The necessary and sufficient conditions for some tensors that satisfy the generalized trirecurrence property have been studied in GβH-TRF_n. Also, we obtained Berwald's covariant derivative of third order for different tensors are non - vanishing.

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