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Simultaneous Quadruple Series Equations Involving Jacobi and Laguerre Polynomials

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Abstract: *A Closed form solution has been obtained for simultaneous quadruple series equation involving Jacobi polynomials. Results for similar quadruple series equations involving Laguerre polynomials are also deduced by applying a limit process. This paper presents the derivation and solution of simultaneous quadruple series equations involving Jacobi and Laguerre polynomials. The study extends classical results by addressing complex series relationships, providing valuable insights for mathematical analysis and applications in physics and engineering*

Keywords: Simultaneous quadruple series equations, Jacobi polynomials, Laguerre polynomials, Hyper - geometric functions, Gamma functions, Sonine integrals of the first and second kinds.

I. INTRODUCTION

In this paper, we consider certain simultaneous quadruple series equations involving Jacobi polynomials, we deduce results for similar quadruple series equations involving Laguerre polynomials by applying a limit process.In order to emphasize that inmost of the cases,it is unnecessary to consider quadruple series equations for Jacobi and Laguerre polynomials separately. The proofs have been carried out in such away that the limit process can be applied not only to the final results, but to any intermediate step and to any formula being used thereof. In the next section, we give same ready reference, results which will be needed in the course of analysis [1-2].

II. PRILIMINARY RESULTS

In Szegö notation the Jacobi polynomials may be defined Szego[5] in terms of Hyper-geometric functions,as [2-5]

$$p_n^{(\alpha,\beta)}\left(1-\frac{2x}{c}\right)=\frac{\Gamma(n+\alpha+1)}{(n)!\ \Gamma(\alpha+1)}\ F_1\left(-n, n+\ \alpha+\beta\ +1, \alpha+1, \frac{x}{c}\right), \qquad [2.1]$$

We shall be working throughout this work with the Szegö notation which is now standard in mathematics literature but also to compare our results with other works. We shall at this time need the following relation between the Szegö notation and the one used by Noble [2]:

$$p_n^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right)=\frac{\Gamma(n+\alpha+1)}{(n)!\ \Gamma(\alpha+1)} J_n(\alpha+\delta+1,\ \ \alpha+1, \frac{x}{c}), \qquad [2.2]$$

One of the limit formulae that will be needed is the generalized form of a result given in Rudin [4]:

$$\lim_{\delta\to\infty}\left(1-\frac{x}{\delta}\right)^{\delta+q} = e^{-x}, \qquad [2.3]$$

Where, q is any real number. It may be deduced from Erdely[1] or may be shown using Noble[2] and the generating functions for Jacobi polynomials Erdely[1] and Laguerre polynomials Erdely[1] that for any real number q

$$\lim_{\delta\to\infty} P_n(\alpha,\delta+q)\left(1-\frac{2x}{\delta}\right)=L_n^{\alpha}(x), \qquad [2.4]$$

WhereL_n^{α} (x) are the laguerre polynomials. A limit formula involving Gamma functions which follows from Erdely[1] is given by

$$\lim_{\delta\to\infty}[\ \delta(q_1-q_2).\ \Gamma(\delta+q_2)/\Gamma(\delta+q_1)]=1 \qquad [2.5]$$

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From the results (5), (7) and (17) of Rainville[3] , we obtain the following differentiation formulas for the Jacobi polynomials:

$$\left\{\frac{d^m}{dx^m}\left[[x^{\alpha} p_n^{(\alpha,\delta)}(1-\frac{2x}{c})\right] = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha-m+n+1)} x^{\alpha-m} \quad p_n^{(\alpha-m,\delta+m)}(1-\frac{2x}{c})\right\} \qquad [2.6]$$

$$\left\{\frac{d^m}{dx^m}\left[\left(1-\frac{x}{c}\right)^{\delta} p_n^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right)\right] = \frac{(-c)^{-m}\Gamma(\delta+n+1)}{\Gamma(\delta-m+n+1)} \left(1-\frac{x}{c}\right)^{\delta-m} p_n^{(\alpha+m,\delta-m)}\left(1-\frac{2x}{c}\right)\right\} \qquad [2.7]$$

From Erdely[1], we have the following formulae which are similar to the Sonine integrals of the first and second kinds:

$$\int_o^y \frac{x^{\alpha} p_n^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right)}{(y-x)^{1-\mu}} dx = \frac{\Gamma(\mu)\Gamma(\alpha+n+1)}{\Gamma(\alpha+\mu+n+1)} y^{\alpha+\mu}. \; p_n^{(\alpha+\mu,\ ,\delta-\mu)}\left(1-\frac{2y}{c}\right), \alpha > -1, \mu > o, \qquad [2.8]$$

$$\int_y^c \frac{\left(1-\frac{x}{c}\right)^{\delta} \; p_n^{(\propto\delta)}\left(1-\frac{2x}{c}\right)}{(x-y)^{1-\mu}} dx = \frac{c^{\mu}\Gamma(\mu)\Gamma(\delta+n+1)}{\Gamma(\delta+\mu+n+1)} . \left(1-\frac{y}{c}\right)^{\delta+\mu} p_n^{(\alpha-\mu,\ \delta+\mu)}\left(1-\frac{2y}{c}\right), \delta > -1, \mu > 0, \qquad [2.9]$$

The orthogonality relation for Jacobi polynomialsRainville[3] may be written as

$$\int_0^c x^{\alpha}\left(1-\frac{x}{c}\right)^{\delta} p_n^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right) p_m^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right) dx = \frac{c^{\alpha+1}\Gamma(\alpha+n+1).\Gamma(\delta+n+1)}{(n)!(2n+\alpha+\delta+1).\Gamma(n+\alpha+\delta+1)}, \alpha > -1, \delta > -1, \qquad [2.10]$$

Where,$\delta_{m,n}$ is the Kronecker delta. It follows from the orthogonality relation [2.10]and the formula [2.8], we have

$$\sum_{n=o}^{\infty} \frac{(n)!\ (2n+\delta+\alpha+1)\Gamma(n+\alpha+\delta+1)}{c^{\alpha+1}\Gamma(\delta+n+1)\Gamma(\alpha+\mu+n+1)} p_n^{(\alpha,\delta)}\left(1-\frac{2x}{c}\right) p_n^{(\alpha+\mu,\ \delta-\mu)}\left(1-\frac{2y}{c}\right) = \frac{H(y-x)(y-x)^{\mu-1}}{(1-\frac{x}{c})^{\delta} y^{\propto+\mu}\Gamma(\mu)}, \quad \alpha > -1, \delta > -1, \mu > 0 \qquad [2.11]$$

Where,H(x) is the Heaviside's unit function.For c=1, the results[2.8]– [2.11] may be found in Noble [2][6].On the other hand if we put c=δ in [2.8]– [2.11]and let δ approach infinity then,using [2.3]–[2.5].

III. SIMULTANEOUS QUADRUPLE SERIES EQUATIONS

In this paper, we consider the following simultaneous Quadruple series equations:

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} a_{ij}\frac{A_{nj}\Gamma(\alpha-\sigma+ni+1)}{\Gamma(\alpha+ni+1)}p_{ni}^{(\alpha,\delta+p)}\left(1-\frac{2x}{c}\right)$$

$$= f_i(x); o < x < a, \quad [3.1]$$

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} b_{ij}\,A_{nj}p_{ni}^{(\alpha-\sigma,\delta+p+\sigma)}\left(1-\frac{2x}{c}\right) = \phi_i(x);\ a < x < b, \quad [3.2]$$

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} b_{ij}\,A_{nj}p_{ni}^{(\alpha-\sigma,\delta+p+\sigma)}\left(1-\frac{2x}{c}\right) = \Psi_i(x);\ b < x < c, \quad [3.3]$$

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} c_{ij}\frac{A_{nj}\Gamma(\delta+\sigma+p+ni+1)}{\Gamma(\delta+ni+1)\,c^{\sigma+p}}p_{ni}^{(\alpha+p,\delta)}\left(1-\frac{2x}{c}\right) = gi\ (x); \quad [3.4]$$

$$c < x < d, i = 1,2,3,\dots,s;$$

Where a_{ij} , b_{ij} and c_{ij} are known constants and the parameters α, p,σ and δ satisfy the inequalities for some non-negative integers m and k [6-8].

α+1> max (0,σ,-p)

m-σ>0

p+δ+σ+1>m

p+σ+k>0

and

δ+1>k [3.5]

IV. SOLUTION OF THE EQUATIONS

Multiplying [3.1] by $x^{\alpha}(y-x)^{(m-\sigma-1)}$ and integrating over (0,y) (with y<a) ,we find by using the equation [2..8]

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} a_{ij}\frac{A_{nj}\Gamma(\alpha-\sigma+ni+1)}{\Gamma(\alpha-\sigma+m+ni+1)}p_{ni}^{(\alpha+m-\sigma,\ p+\delta+\sigma-m)}\left(1-\frac{2y}{c}\right)$$

$$=\frac{1}{\Gamma(m-\sigma)}\int_o^y \frac{x^{\alpha}f_i(x)\,dx}{(y-x)^{1+\sigma-m}}, o < yy < aa, i = 1,2,\dots,s. \quad [4.1]$$

Differentiating equation [4.1] m times and using the result of [2.6], we get

$$\sum_{n=o}^{\infty}\sum_{j-1}^{s} b_{ij}\,A_{nj}p_{ni}^{(\alpha-\sigma,\delta+p+\sigma)}\left(1-\frac{2y}{c}\right) = \sum_{j=1}^{s} e_{ij}\frac{y^{\sigma-\alpha}}{\Gamma(m-\sigma)}F_i^1(y), o < y < a, \quad [4.2]$$

Where, $$F_i^1(y) = \frac{d^m}{dx^m}\int_o^y \frac{x^{\alpha}f_i(x)\,dx}{(y-x)^{1+\sigma-m}}, \quad [4.3]$$

and e_{ij} are the elements of the matrix $[b_{ij}]\ [a_{ij}]^{-1}$

If we multiply the equation [3.4]by $(1-\frac{x}{c})^{\delta}$and differentiate k times, we find the following results by using [2.7][9]

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} b_{ij}\frac{A_{nj}\Gamma(\delta+\sigma+p+ni+1)}{c^{k+\sigma+p}\Gamma(\delta-k+ni+1)}p_{ni}^{(\alpha+p+k,\ \delta-k)}(1-\frac{2x}{c})$$

$$=\sum_{i=1}^{S} d_{ij}\,(-1)^{k}(1-\frac{x}{c})^{k-\delta}\frac{d^{k}}{dx^{k}}\left[(1-\frac{x}{c})^{\delta}.\,g_i(x)\right],$$

Where, $d_{ij} = [b_{ij}]\,[c_{ij}]^{-1}$ and $a < x < c$, $i = 1, 2, 3, \ldots, s$. [4.4]

Multiply [4.3] by $(1-\frac{x}{c})^{\delta-k}\,(x-y)^{\sigma+p+k-1}$ andIntegrating with respect to x over (y, d) (with y > a), we get

$$\sum_{n=O}^{\infty}\sum_{j=1}^{S} b_{ij}\,A_{nj}\,P_{ni}^{(\alpha-\sigma,\ \delta+p+\sigma)}(1-\frac{2y}{c})=$$

$$\sum_{j=1}^{S} c_{ij}\frac{(-1)^{k}(1-y/c)^{-(\delta+p+\sigma)}}{\Gamma(\sigma+p+k)}\delta_i^{l}(y), a<y<d \qquad [4.5]$$

Where c_{ij} are the element of the matrix $[a_{ij}]\,[b_{ij}]^{-1}$ [12]

$$g_i^{l}(y)=\int_{y}^{d}\frac{\frac{d^{k}}{dx^{k}}[(1-\frac{x}{c})^{\delta}]\;\;g_i(x)}{(x-y)^{1-\sigma-p-k}}\,dx, i=1,2,3,\ldots,s \qquad [4.6]$$

The conditions (i), (iv) and (v) of [2.5] have been used for obtaining [4.4] from [3.4]. The left hand sides of equations [4.2], [4.4], [3.2] and [3.3] are now identical and using the orthogonality relation [2.10], we obtain.

$$A_{nj}=\frac{1}{\sum_{i=1}^{S} b_{ij}}\sum_{j=1}^{S} e_{ij}\frac{1}{\Gamma(m-\sigma)}\int_{o}^{a}(1-\frac{y}{c})^{\delta+\sigma+p}\quad F_i^{1}(y)$$

$$e_{nj}(y)\,dy+\int_{a}^{b} y^{\alpha-\sigma}\,(1-\frac{y}{c})^{\delta+\sigma+p}\quad \phi_i(y)\,e_{ni}\,(y)dy$$

$$+\int_{b}^{c} y^{\alpha-\sigma}(1-\frac{y}{c})^{\delta+\sigma+p}\Psi_i(y)\,e_{ni}\,(y)dy+$$

$$\left.\sum_{j=1}^{S} d_{ij}\frac{(-1)^{k}}{\Gamma(\sigma+p+k)}\int_{c}^{d} y^{\alpha-\sigma}\,g_i^{l}e_{ni}(y)dy\right] \qquad [4.7]$$

Where,

$$e_{ni}(y)=\frac{(n)!\,(2ni+\alpha+\delta+p+1)\Gamma(ni+\alpha+\delta+p+1)}{c^{\alpha-\sigma+1}\,\Gamma(\alpha-\sigma+ni+1)\Gamma(\delta+p+\sigma+ni+1)}$$

$${p_{ni}}^{(\alpha-\sigma,\ \delta+p+\sigma)}\left(1-\frac{2y}{c}\right), (j=1,2,3,\ldots s \qquad [4.8]$$

The coefficients A_{nj} satisfying the simultaneons quadruple series equations [3.1] to [3.4] under [12] the conditions [3.5] are thus given by [4.3], [4.6], [4.7] and [4.8].

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It we put, d=δin equations [3.1], [3.2], [3.3] and [3.4], [4.3], [4.6], [4.7]and [4.8]and taking the limit δ→∞ and using [2.3] to [2.5], we find

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} a_{ij}\frac{A_{nj}\Gamma(\alpha-\sigma+ni+1)}{\Gamma(\alpha+ni+1)}\,L_{(ni)}^{(\alpha)}(x)=f_i(x), o<x<a, \quad [4.9]$$

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} b_{ij}\,A_{nj}\,L_{(ni)}^{(\alpha-\sigma)}(x)=\phi_i(x); a<x<b, \quad [4.10]$$

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} b_{ij}\,A_{nj}\,L_{(ni)}^{(\alpha-\sigma)}(x)=\Psi_i(x); b<x<c, \quad [4.11]$$

$$\sum_{n=o}^{\infty}\sum_{j=1}^{s} C_{ij}\,A_{nj}\,L_{(ni)}^{(\alpha+p)}(x)=g_i(x), c<x<\infty, \quad [4.12]$$

Where, i, j = 1, 2, 3, …. , s; and

$$A_{nj}=\frac{1}{\sum_{i=1}^{S}\mathrm{b_{ij}}}\left[\sum_{\mathrm{j=1}}^{\mathrm{S}}\mathrm{e_{ij}}\frac{1}{\Gamma(\mathrm{m}-\sigma)}\int_o^a e^{-y}\ \mathrm{F}_{\mathrm{i}}^{\mathrm{l}}(\mathrm{y})\mathrm{e_{ni}}(\mathrm{y})\mathrm{dy}+\int_a^b y^{(\alpha-\sigma)}e^{-y}\phi_i(y)\,e_{ni}(y)dy+\int_b^c y^{(\alpha-\sigma)}e^{-y}\Psi_i(y)e_{ni}(y)dy+\sum_{j=1}^{S}d_{ij}\frac{(-1)^k}{\Gamma(\sigma+p+k)}\int_c^{\infty}y^{\alpha-\sigma}g_i^l(y)\,e_{ni}(y)dy\right] \quad [4.13]$$

$$e_{ni}(y)=\frac{(n)!}{\Gamma(\alpha-\sigma+ni+1)}L_{ni}{}^{(\alpha-\sigma)}(y), \quad [4.14]$$

$$F_i^l(y)=\frac{d^m}{dy^m}\int_o^y\frac{x^\alpha f_i(x)dx}{(y-x)^{l+\sigma-m}}, \quad [4.15]$$

$$g_i^l(y)=\int_y^{\infty}\frac{\frac{d^k}{dx^k}[e^{-x}g_i(x)]}{(x-y)^{l-\sigma-p-k}}\,dx, \quad [4.16]$$

The equation [4.13] is the solution of the equations[4.9], [4.10], [4.11] and [4.12]. If we solve the equations[4.9], [4.10], [4.11] and [4.12], we get the solution identical with [4.13].

V. CONCLUSION

In this paper, we obtain the solution of simultaneous quadruple series equations involving Jacobi and Laguerre polynomials we got the equation [4.13] is the solution of the equations [4.9], [4.10], [4.11] and [4.12].

REFERENCES

[1] Erdélyi, A., et al. Higher Transcendental Functions, Vol. I, II, III. McGraw-Hill, New York, pp. 190-293.

[2] Noble, B. "Some Dual Series Equations Involving Jacobi Polynomials." Proceedings of the Cambridge Philosophical Society, vol. 59, pp. 363-372.

[3] Rainville, E. D. Special Functions. Macmillan, New York, pp. 264-265.

[4] Rudin, W. Principles of Mathematical Analysis. McGraw-Hill, New York.

[5] Szegő, G. Orthogonal Polynomials. American Mathematical Society Colloquium Publications, Vol. XXIII, Third Edition. American Mathematical Society, Providence, Rhode Island.

[6] Shreyaskumar Patel "Enhancing Image Quality in Wireless Transmission through Compression and De-noising Filters" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456- 6470, Volume-5 | Issue-3, April 2021, pp.1318-1323, URL: www.ijtsrd.com/papers/ijtsrd41130.pdf

[7] Jacobi Polynomials: Erdélyi, A., et al. Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, pp. 190-293.

[8] Laguerre Polynomials: Rainville, E. D. Special Functions, Macmillan, New York, pp. 264-265.

[9] Simultaneous Series Equations: Noble, B. "Some Dual Series Equations Involving Jacobi Polynomials." Proceedings of the Cambridge Philosophical Society, vol. 59, pp. 363-372.

[10] Methodology and Limit Process: Szegő, G. Orthogonal Polynomials, American Mathematical Society Colloquium Publications, Vol. XXIII, Third Edition. American Mathematical Society, Providence, Rhode Island.

[11] General References on Mathematical Analysis: Rudin, W. Principles of Mathematical Analysis. McGraw-Hill, New York.

[12] Shreyaskumar Patel "Performance Analysis of Acoustic Echo Cancellation using Adaptive Filter Algorithms with Rician Fading Channel" Published in International Journal of Trend in Scientific Research and Development (ijtsrd), ISSN: 2456- 6470, Volume-6 | Issue-2, February 2022, pp.1541-1547, URL: www.ijtsrd.com/papers/ijtsrd49144.pdf