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Use of Similarity Invariants for Identifying Non Similar Matrices

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Abstract: *Matrix similarity is a cornerstone concept in linear algebra that identifies when two matrices represent the same linear transformation under different bases. Demonstrating similarity is relatively direct using change-of-basis techniques, whereas proving non-similarity requires careful reasoning. This paper develops a structured framework to establish non-similarity of matrices by leveraging invariants such as characteristic polynomials, minimal polynomials, eigenvalues, canonical forms, and rank sequences. Illustrative examples are provided to demonstrate how these invariants distinguish matrices that cannot be transformed into one another*

Keywords: Non-Similarity, Invariants, Minimal and Characteristic polynomial etc

I. INTRODUCTION

Two square matrices A and B over a field F are called similar if there exists an invertible matrix P such that $B = P^{-1}AP$. This relationship groups matrices into equivalence classes where members share the same structural properties. In many applications, such as systems theory, quantum mechanics, and numerical linear algebra, distinguishing whether matrices are similar is crucial. However, when matrices are not similar, the challenge is to rigorously demonstrate this fact. The aim of this paper is to provide a systematic methodology for proving that matrices are not similar.

II. THEORETICAL BACKGROUND

Similarity preserves certain structural properties of matrices. Consequently, if two matrices differ in at least one invariant, they cannot be similar. The following properties are fundamental invariants under similarity:

1. **Characteristic Polynomial** – Similar matrices must have identical characteristic polynomials.
2. **Minimal Polynomial** – The minimal polynomial is also preserved under similarity.
3. **Jordan Canonical Form –** The sizes and arrangement of Jordan blocks remain unchanged.
4. **Rank and Nullity of Powers –** The nullity of A^k is invariant for all $k \geq 1$
5. **Determinant and Trace –** These values remain the same for similar matrices.
6. **Rational Canonical Form –** Provides a complete invariant classification over any field.

III. METHODS TO PROVE NON-SIMILARITY

3.1 Determinant Comparison

If the determinant of two matrices are different then they are not similar. **Example:** $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$.Here determinant of A is 18 and of B is 16 hence they are not similar matrices.

3.2 Trace Comparison

If the trace of two matrices are different then they are not similar.

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$. Here trace of A is 2 and of B is 3 hence they are not similar.

3.3 Eigenvalue Comparison

The simplest test involves comparing eigenvalues. If two matrices have different eigenvalues, then they cannot be similar.

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Example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$. Matrix A has eigenvalues $\{1, 2\}$, while B has eigenvalues $\{3, 4\}$. Since the eigenvalue sets differ, A and B are not similar.

3.4 Minimal polynomial differences

Even if matrices share the same eigenvalues, their minimal polynomials may differ, which prevents similarity.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Both matrices have the eigenvalue 2. However, the minimal polynomial of A is $(x-2)^2$ while the minimal polynomial of B is $(x-2)$. Since the minimal polynomials are not identical, A and B are not similar.

3.5 Jordan Canonical Form

Jordan canonical form captures the geometric and algebraic multiplicities of eigenvalues. Two matrices with identical eigenvalues but different Jordan block structures are not similar.

Example: A Jordan block J = $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not similar to the diagonal matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Observe that Both share eigenvalue 2, but the Jordan block structure differs.

3.6 Rank of Powers

The ranks and nullities of powers of a matrix are preserved under similarity. Thus, by examining the sequence of nullities of A, A^2, A^3 etc., we can determine whether two matrices are not similar.

Example: Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Matrix A has nullities: null(A) = 1, null(A^2) = 2. Matrix B has nullities: null(B) = 2, null(B^2) = 2. The sequences differ, therefore A and B are not similar.

3.7 Rational Canonical form

When Jordan form is inconvenient or not defined over the base field, the rational canonical form provides a unique classification of similarity classes. Distinct rational canonical forms imply non-similarity.

Consider matrices over any field of characteristic $\neq 2$. $A =$ companion of $(x-1)^2$. i.e. $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ and B is identity matrix of order 2. Rational canonical forms (RCF) of A is a single 2×2 companion block with invariant factors $\{(x-1)^2\}$ and the RCF of B are two 1×1 blocks (just 1,1) with invariant factors $\{(x-1), (x-1)\}$. Both have the same characteristic polynomial $(x-1)^2$, but the invariant factors (hence their rational canonical forms) are different. Since RCF is a complete similarity invariant, A and B are not similar. (Equivalently observe that minimal polynomials are different so they can't be similar.)

IV. CASE STUDIES

Case 1: Two matrices may share the same characteristic polynomial but differ in minimal polynomials. This occurs in defective versus diagonalizable matrices.

Case 2: Matrices with the same eigenvalues but different Jordan block structures are not similar. This highlights the importance of geometric multiplicity.

Case 3: Matrices with equal eigenvalues and polynomials but distinct nullity progressions reveal non-similarity through kernel dimension analysis.

V. APPLICATIONS

The ability to distinguish non-similar matrices has applications across several fields:

- In control theory, similarity determines whether two system representations are equivalent. Non-similarity demonstrates fundamentally distinct system dynamics.

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- In quantum mechanics, operators representing physical systems may be non-similar, reflecting different measurement outcomes.
- In computer science, distinguishing non-similar matrices plays a role in graph isomorphism problems and complexity theory.

VI. CONCLUSION

Proving that two matrices are not similar is a rigorous exercise in identifying structural invariants that differ between them. The methods explored—comparison of eigenvalues, minimal polynomials, Jordan canonical forms, rational canonical forms, and rank sequences—provide a systematic toolkit for demonstrating non-similarity. By applying these techniques to illustrative examples, one can confidently conclude when matrices belong to different similarity classes.

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