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Complex Analysis, Imaginary Numbers in Real World

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Abstract: *This article explores introductory concepts related to complex numbers. The complexity of a society often parallels the increasing demand for mathematical understanding. Here, we delve into the nth roots and solutions of equations of the form z^n = 1. Complex analysis is a pivotal subject for students in engineering, computing, the physical sciences, and mathematics. Over the past four centuries, complex systems have been subject to intensified study, becoming an accepted mathematical framework for the sake of representation or choice. With the emergence of more abstract proofs, mathematicians gained confidence in developing techniques for solving complex systems. Today, the study of complex numbers has evolved into an independent subject known as complex analysis. This advanced exploration of complex numbers, along with the expansion and simplification of proofs, has opened up new and expansive perspectives for approaching various branches of mathematics.*

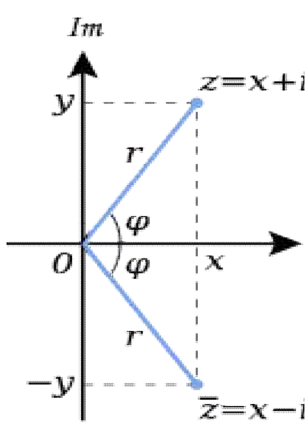
Keywords: Complex Numbers, Analytical functions, Series, Sequence, Convergence, Residues

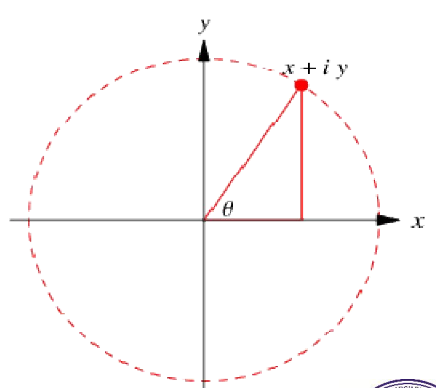
I. INTRODUCTION

In the year when the renowned Swiss mathematician Leonhard Euler made significant contributions to the field, he introduced a set of principles that laid the foundation for modern mathematics. Among these principles, he introduced the concept of a number 'i,' which he named "Iota." Euler defined 'i' as the imaginary unit of a complex number, and its square equated to -1. The introductionof 'i' allowed for the interpretation of the square root of a negative number as a product of a real number with 'i,' denoted as x = ± i.

A complex number is any number expressed in the form of x + iy, where x and y are real numbers, and 'i' represents the square root of -1. It is essentially a combination of a real number and an imaginary number. A complex number is typically denoted by the letter 'z,' such as z = x + iy. Here, 'x' is referred to as the real part of z and is denoted as Re(x + iy), while 'y' is the imaginary part, represented as Im(x + iy). In cases where x = 0 and y ≠ 0, the complex number becomes purely imaginary, represented as iy. Examplesof purely imaginary numbers include -4i, 1/2i, 6i, 5i, and πi.

The complex plane is illustrated below for reference. In a complex number z = x + iy, the real part is denoted as 'x,' and the imaginary part is 'y.' The set of all complex numbers, denoted as C, includes numbers that can be expressed as z = x + iy, where x and y are real numbers. In this set, 'i' is defined asthe square root of -1, with the property that 'i^2' equals 1. Thus, 'i^n' belongs to the set {-1, 1, i, 1} for all 'n' in the set of integers





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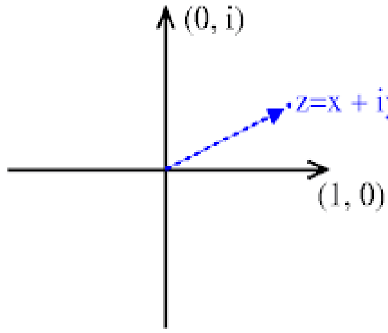

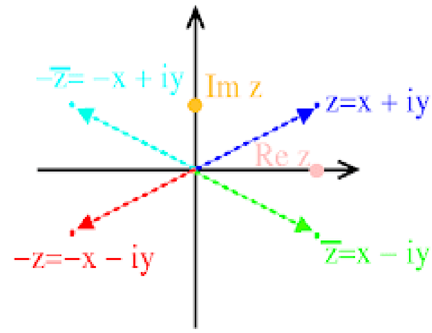

.

If x = 0, the number is said to be Purely Imaginary, and if y = 0, the number is Real…Note that Zero is the only number which is both Real and Imaginary Number.. x and y are respectively the real and Imaginary parts of the complex number x + iy, denoted by z i.e z= x + iy…Other standard notation weuse to denote a Complex Number is w + u+iv… The Real and Imaginary parts of a complex number are denoted by Re z, and Im z…. A Complex Conjugate of a Complex Number z = x + iy is x-iy and is denoted by $\bar{z}$ = x – iy .

…The Number is real when conjugate of z= $\bar{z}$.…The Real and Imaginary parts of a complex number canbe expressed in terms of the complex number and its conjugate

The Modulus of a Complex Number z= x + iy is given by

$$|z| = \sqrt{\sqrt{x^2}} +$$

$$|z|^2 = x^2 + y^2 = (Re\ z)^2 + (Im\ z)^2 \geq (Re\ z)^2$$

In Geometric Interpretation we use Polar Coordinates..If the polar coordinates of (x,y) are (r, θ), then x=r cos θ and y = r sin θ, Hence z = x + iy = (Cos θ + i Sin θ) ..r ≥ 0, r = |z| = $\sqrt{x^2 + y^2}$ ..The Polar angle φ, is called the is called Argument (or Amplitude) of the Complex Number..

Binomial Equation :--Now the n^{th} power of the Complex Number z = r (Cos θ + i Sin θ) , is given by

$z^n = r^n$ (Cos n θ – i Sin n θ).

The Formula is Trivially Valid for z= o, and since

$z^{-1} = r^{-1}$ (Cos θ – i Sin θ) = r^{-1} [Cos(-θ) + i Sin (-θ)], it holds also for when n is a negative integer…

For r =1 , we have De Movers Theorem

$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

It may be noted that n^{th} root of any complex number z≠ 0, have same Modulus andtheir Arguments are Equally Spaced..

In the Geometry of a Complex Number we know the equation of a Circle, with Center at "a" and Radius "r" is given by $|z - a| = r$, .. $z = re^{i\theta}$ is the Parametric representation of a Circle with Radius "r"….

An Inequality $|z - a| < r$ describes the Inside of the Circle….

In Algebraic Form it can be Written as (z-a) ($\bar{\ }$-$\bar{a}$) = r^2……

The Connection between , Cos φ. and Sin φ… is given by Euler's Theorem as

$e^{i\theta} = \cos\theta + i \sin\theta$

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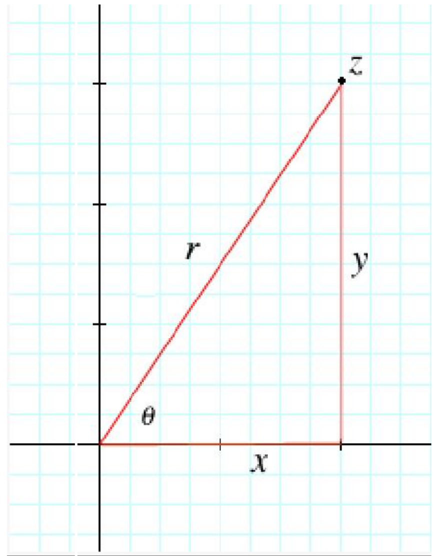

Note that the complex number cosθ + i sinθhas absolute value 1 since cos²θ + sin²θ = 1 for any angle θThus, every complex number z is the product of a real number |z| and a complex number cos θ+ i sin . θ

A Straight Line in a complex plane:- is given by a Parametric Equation z= a+ bt, where a and b are complex numbers and b ≠ 0, "t", is the parameter. The parameter is a Real Value..

$\{Z = a + bt : t \in R\}$

Continuous Function :- A function f(z) is continuous at a point at "a" if and only if

$$\lim_{x \to a} (x) = f(a), \ldots.$$

If f(x) is continuous, then Re f(x), Im f(x) and $|f(x)|$ are also continuous

Derivative of a Complex Number :- **Let w = f(z)**..If the limit is z⟶a, then the derivative f' (z) is defined by :-

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a},$$ Provided the limit exists. If the limit exists we say f is differentiable at "a". and also then only f is analytic at a...

Analytic Funtion:- A function f(z) is an Analytic function, if it has a Complex Derivative f' (z)

i.e $$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z) - f(z)}{\Delta z} = \lim_{z \to 0} \frac{f(z) - f(z0)}{z - z0}$$ --

...The limit has to exist and be the same, no matter how you approach "z_0 or just 0"

Partial Derivatives as Limits :- Let us remind the Partial Differential equations before we proceed for the Cauchy Riemann's Equations:-

If u(x, y) is a function of two complex variables then the partial derivatives of u are defined as

$$\frac{\partial u}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$, when "y" is constant

& when "x" is constant then-

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$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}$$

Series of a Complex Number :-

The summation of a series $a_1 + a_2 + a_3 + a_4 + \text{---------} + a_n + \text{-----}$, Denoted $by: -$

$\sum_{n=1}^{\infty} an = a_1 + a_2 + a_3 + a_4 + \text{---------} + a_n + \text{----}$, is called an infinite series of a complex number...

Now we write $s_n = a_1 + a_2 + \text{----------------} + a_n$, Here s_n is called the n^{th} partial sum of the series $\sum_{n=1}^{\infty} an$....

The Sequence $\{s_n\}$ is called the Sequence of partial sum of $\sum_{n=1}^{\infty} an$..

A sequence is called a Cauchy Sequence if it satisfiers the following condition :-

If given any $\varepsilon > 0$ $\exists$ n_0 : $|an - am| < \varepsilon$, whenever $n \geq n_0$, and also $m \geq n_0$.....

A sequence is convergent if and only if it's a Cauchy sequence….

Cauchy's Criterion for uniform convergence:-

$\sum_{n=1}^{\infty}$ an is convergent if and only if to each $\varepsilon > 0$ $\exists$ $n_0 \in N$: $| a_{n+1} + a_{n+2} + \text{---------} + a_{n+p} | < \varepsilon$ $\forall$ $n \geq n_0$, $p > 1$

The Sequence $\{f_n(z)\}$ Converges Uniformly on E, if and only if to each $\varepsilon > 0$, there exists an n_0 such that $| f_m - f_n | < \varepsilon$ $\forall$ $m,n \geq n_0$ and all $Z \in E$...

PROOF:- Suppose $\{f_n(z)\}$ converges uniformly on f(z) say on E. Let $\varepsilon > 0$ be given.... $\exists$ $n_0 \in N$, Such that $|f_n(z) - f_m(z)| = | f_m(z) - f(z) + f(z) - f_n(z)| \leq | f_n(z) - f(z) | + |f_m(z) - f(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\therefore | f_m(z) - f_n(z)| < \epsilon$ $\forall$ $m,n \geq n_0$ $\forall$ $z \in E$...Conversely suppose for $\epsilon > 0$ $\exists$ n_0 such that $|f_m(z) - f_n(z)| < \epsilon$ $\forall$ $m,n \geq n_0$ $\forall$ $z \in E$

$\Rightarrow \{ f_n(z)\}$ is a Cauchy Sequence $\Rightarrow \{ f_n(z)\}$ is convergent and the convergence is Uniform as n_0 depends on ϵ..... HENCE PROVED,,,

Power Series :- A formal sum of the form $a_0 + a_1 + a_2 + \text{-------} + a_n + \text{-----}$, where the coefficient a_n and the variable z are Complex Numbers, is called a Power Series:- Denoted as

$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 + a_2 + \text{-------} + a_n + \text{-----}$,....

$\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called the Powere Series w.r.to Center z_0....

For Every Power Series $\sum_{n=0}^{\infty} a_n z^n$ there exists a number R : $0 \leq R \leq \infty$,. Called the Radius of Convergence ,with the following properties

1) The series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for every z with $|z| < R$, If $0 \leq \rho \leq R$, the convergence is uniform for $|z| \leq \rho$
2) If $|z| > R$, the terms of the series $\sum_{n=0}^{\infty} a_n z^n$ get bounded and hence the series is Divergent
3) If $|z| > R$, the sum of the series is an analytic function .Then Derivative can be obtained by term wise differentiation and the derived series has the same Radius of Convergence

This is called **Abel's Theorem** which gives the information about the circle of convergence of a Power series..

Now if $\sum_{n=0}^{\infty} a_n$ converges then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ thends to f(1) as z approaches 1 in such a way that $\frac{|1-z|}{(1-|z|)}$ remains bounded.......This is called **Abel's Limit Theorem**...for power series that converges at a Point on the circle of Convergence....

1) Cauchy Riemann's Equations:-

THEOREM:- If $w = f(z)$ is a Complex valued Analytic Function then :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{---} \ (1)$$

The Eq^ns (1) are called Cauchy- Riemann's Equations

PROOF:- **If** w= f(z) is a C.V Analytic Function & $f'(z)$ exists, whenever $f(z)$ is defined

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i.e $f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$ exists.

In other words the Quotient $\frac{f(z+h) - f(z)}{h}$ should approach the same limit regardless of the way in which h approaches zero...

Let h→0 through Real Values, then :- $\frac{f(z+h) - f(z)}{h} =$

$$= \left[\frac{u(x+h,y) - u(x,y)}{h}\right] + i\left[\frac{v(x+h,y) - v(x,y)}{h}\right]$$

i.e $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} =$

$= U_x - i V_x$ ....(2)

Now let h→0 through Imaginary Values:- Say h = ik , k ∈ R, then-

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{[u(x,y+k) - u(x,y)]}{ik} + i \lim_{k\to 0} \left[\frac{v(x,y+k) - v(x,y)}{ik}\right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -1\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$= i\,U_y + V_y$ ....(3)

Comparing Equations (2) & (3) we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

& $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ -------------Equations (4) C.R Equations..

The Real and Imaginary parts should satisfy C.R Equations...

NOTE :-From the above Equations we have several Formulas for $f'(z)$, namely:-

f '(z) = Ux + i Vx = - i Uy + VY = Ux + i (-Uy)

= Ux - i Uy ∵ Uy =-Vx

= - i (- Vx) + Vy = i Vx + Vy ∵ Uy =-Vx

Moreover

$$|f'|^2 = \left\{\frac{\partial u}{\partial x}\right\}^2 + \left\{\frac{\partial u}{\partial y}\right\}^2$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \quad \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$= J\left[\frac{u,v}{x,y}\right]$ The Jecobion of u,v w.r.to x,y.

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Now The Derivative of an Analytical function is again an Analytical function. This implies that u,v will have continuous partial Derivatives of all orders.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

A function u(x,y) & v(x,y) which satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ is called a Harmonic function…

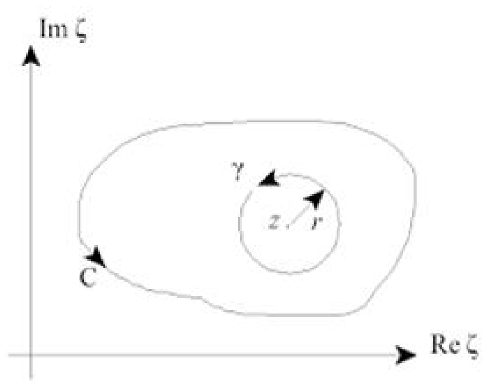
2) Cauchy's Integral Theorem / Cauchy's Theorem for an Analytic Function

Theorem:-If f (z) is a Complex Valued Analytical Function in a Region ℂ, i.e f '(z) is continuous at all points , then

$$\oint_\gamma (z) = 0 \qquad \text{-------------------------------------}(1)$$

For every cycle γ which is homologous to Zero in ℂ..

Proof: -Consider the figure:- Here we have a closed curve γ , an analytic function f (z), within and on a closed curve γ, contained in a region ℂ



We have z = x + iy -------------------------------------- (2)

$\Longrightarrow$ dz = dx + idy -------------------------------------- (3)

Now f(z) = u + iv ------------------------------------ (4)

Now Integrating f(z) in a closed curve γ, we get:-

$$\oint_\gamma (z)dz = \oint_\gamma (u+iv)(dx+idy) \qquad \text{------------------}(5)$$

$$\Longrightarrow \oint_\gamma (z)dz = [udx + iudy + ivdx - vdy] \qquad \text{-----------} (6)$$

Now separating the Real & Imaginary Parts we get:-

$$\oint_\gamma (z)dz = \oint_\gamma (udx - vdy) + i \oint_\gamma (vdx + udy) \qquad \text{--------} (7)$$

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Now from Green's Theorem we have:-

$$\oint_{\gamma} Pdx + Qdy = {}_{C}\int\int\{\frac{\partial Q}{\partial X} - \frac{\partial P}{\partial Y}\}\, dx\, dy$$

Applying Green's Theorem in Equation in Eq^{n} (7), we get

$$\oint_{\gamma} udx - vdy = \int_C\int[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}]\, dx\, dy$$
$$= -\int_C\int[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}]\, dx\, dy$$

$$\& \quad \oint_{\gamma} vdx + udy = \int_C\int[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}]\, dx\, dy$$

$$\therefore \oint_{\gamma} (z)dz = -\int_C\int[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}]\, dx\, dy + i\int_C\int[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}]\, dx\, dy \text{-------- (8)}$$

Now we know that $f(z)$ is an Analytic function completely contained in Region/Domain C, and its Real and Imaginaray parts Satisfy C.R Equations i.e $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Now putting $\frac{\partial u}{\partial x}$ as $\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}$ as $-\frac{\partial v}{\partial x}$ in Eq^{n} (8) , the Terms of the Real and Imaginary Parts in the RHS of the Equation cancels each other and Vanishes Independently & becomes Zero, Showing That

$$\oint_{\gamma} (z) = 0$$

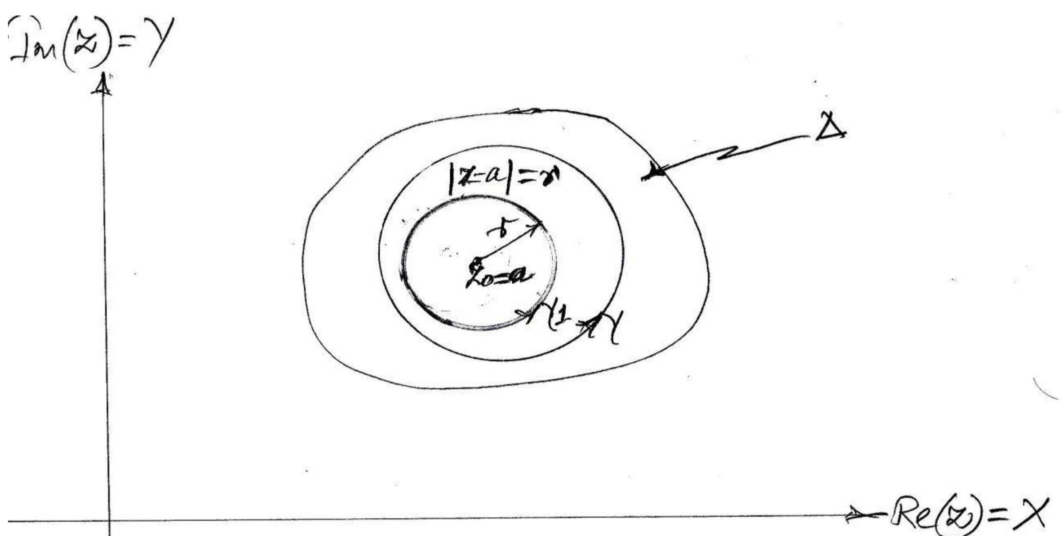
Hence Proved.......

3) Cauchy's Integral Formula

Theorem:- Suppose that $f(z)$ is Analytic in an open Disk Δ, and let γ be a closed curve contained in Δ ... Let "a" be a point inside γ, then :

$$n(\gamma,a) f(a) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(z)}{z-a} dz$$, where n(γ,a) is the index of "a" w.r.to γ... (1)

Proof:-



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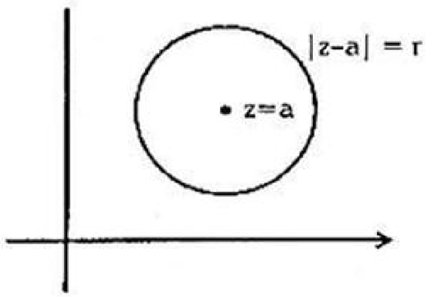
Consider the figure :- We have an open disk Δ, a closed curve γ, and a point "a" inside γ.... Since "a" is ma point within γ, we shall enclose it by a closed curve/ circle γ_1 center at "a" and radius "r"..., such that γ_1 is entirely within γ ..

$r \longrightarrow 0,$
$r > 0$



The function $\frac{f(z)}{z-a}$ is analytic witin and on the boundaries of the annular region between γ & γ_1 (Except on the point "a") in Δ...

Now as a Consequence of Cauchy's Integral Theorem for an Analytic Function, we get

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{\gamma 1} \frac{f(z)}{z-a} \quad \text{--} \quad (2)$$

Now the circle γ_1 (Circle with center at "a" and radius "r"), can be written as:- $|z-a| = r \quad \Rightarrow \quad z - a = re^{i\theta}$

Or $\quad z = a + re^{i\theta} \quad \Rightarrow \quad dz = ire^{i\theta} d\theta, \quad 0 \leq \theta < 2\pi,$

Using the results in the RHS of the Equation (2), we get:-

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{\theta=0}^{2\pi} \frac{(a+re^{i\theta})}{re^{i\theta}} . \ i \ d\theta, \qquad r > 0$$

$$\Rightarrow \oint_{\gamma} \frac{f(z)}{z-a} dz = i \int_0^{2\pi} (a) \qquad r \longrightarrow 0$$

$$\Rightarrow \oint_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\Rightarrow f(a) = \frac{1}{2\pi} \oint \frac{f(z)}{z-a} dz \text{ --------------------------------------- } (3)$$

If multiple loops are made around the point 'a', then the above Equation(3) becomes

$$\mathbf{n(\gamma,a)}.\, f(\mathbf{a}) = \frac{\mathbf{1}}{\mathbf{2\pi i}} \oint_{\gamma} \frac{f(z)}{z-a} \mathbf{dz} \text{ --- } (4$$

where $n(\gamma,a)$. is the index of "a', w.r.to γ... The above equation (4) is called Representation Formula...

In fact "a" can be replaced by any $\xi \in \Delta - \gamma$, so that the above formula becomes:

$$f(\xi) = \frac{1}{2\pi i} \oint_{\gamma} \frac{(z)}{z-\xi} dz$$

This is called the Cauchy's Integral Formula......Proved...

4) <u>Cauchy's Residue Theorem</u>:-

If f(z) is analyticexcept for isolated singulaties a_j in a region Ω, then for any Cycle γ which is homologous to Zero in Ω and that does not pass through any a_j, we have

$$\frac{1}{2\pi i} \oint_{\gamma} (z) = \sum_{j=0}^{m} n(\gamma, a) \, Res_{z=aj} \, f(z)$$

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Proof:- First we assume that the Isolated Singularities are finite in numbers and they are a_1, a_2, a_3, -------, a_m in a region Ω.

Let $\Omega^I = \Omega - \{a1, a2, a3, - - - - - - - - - - -, am\}$

To each a_j there is a $\delta_j > 0$, such that $0 < |z - aj| < \delta_j$ lies in Ω^I

If $\xi_j = \{z: ||z - aj||rj\}$ where $0 < r_j < \delta_j$

Then by the following Theorem { If f(z) has an isolated singularity as "a" then there is an unique complex number R , such that $f(z) - \frac{R}{z-a}$ is the derivative of a single valued analytical functionin the annulus $0 < |z - a| < \delta$ } we get:-

$$Res_{z=aj}\, f(z) = \frac{1}{2\pi i} \oint_{cj} f(z)dz \qquad \text{--------------------------------}(1)$$

Also since $\gamma \sim \sum_{j=1}^{m} (\gamma, a)\, c_j$ with respect to Ω^I

Therefore

$$\oint_{\gamma} f(z)dz = \int_{\sum_{j=1}^{m} n(\gamma,a)} f(z)dz = \sum_{j=1}^{m} n(\gamma, a) \int_{cj} f(z)dz$$

$$\Rightarrow \int_{\gamma} f(z)dz = \int_{\sum_{j=1}^{m} n(\gamma,a)cj} f(z)dz = \sum_{j=1}^{m} n(\gamma, aj) \int_{cj} f(z)dz$$

Which can be written , in view of Equation (1)

$$\int_{\gamma} f(z)dz = \sum_{j=1}^{m} n(\gamma, a).\, 2\pi i\, Res_{\,z=aj}\, f(z)$$

i.e $$\frac{1}{2\pi i} \int_{\gamma} f(z)dz = \sum_{j=1}^{m} (\gamma, a)\, Res_{\,z=aj}\, f(z)$$

Some important notes on Residue of a Complex Number

1) Residue at (z=a) = $\lim_{z-a} (z - a)\, f(z)$
2) Residue at ∞ $= \lim_{z \to \infty} [-z\,(z)]$
3) Residue of pole of order m =

$$\frac{1}{(m-1)!} \lim_{z \to z0} \frac{d^{m-1}}{dz^{m-1}} \{(z - z0^{m})(z)\}$$

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5) MORERA'S THEOREM:-

Suppose f(z) is continuous in a region Ω and $\int_{\gamma} (z)dz = 0$ ∀ closed curves γ in Ω, then f(z) is analytic in Ω…

Proof:-Since f(z) is continuous in Ω, and $\int_{\gamma} (z)dz = 0$ ∀ closed curves γ, we have that f(z) dz is an exact differentiasl…

Hence there is a continuous function F(z) on Ω, such that $F^{I}(z) = f(z)$ ∀ z ∈ Ω

That is, F(z) is analytic on Ω

Fronm the Theorem { If f(z) is an analytic functiondefined on a region Ω then f(z) has derivatyive of all orders}…..

Now we see that $F^{I}(z)$ is also Analytic

That is f(z) is analytic on Ω. Proving the Theorem…

Hence proved..

6) LIOUVILLE'S THEOREM:-

A function f(z) which is analytic in the entire finite complex plane and is bounded, is a constant function..

Proof:- Let $|f(\xi)| \leq M \quad \forall\ \xi \in C$

Fix a ∈ C

Let $\gamma: |\xi - a| = R$

Then $f^{I}(a) = \frac{1}{2\pi I}\int_{\gamma} \frac{f(\xi)}{(\xi-a)^2}\, d\xi$

$\Rightarrow |(a)| \leq \frac{1}{2\pi i}\int \frac{|f(\xi)|}{|\xi-a|^2}\,|d\xi| \leq \frac{1}{2\pi}\frac{M}{R^2}\int_{\gamma} |d\xi| = \frac{1}{2\pi}\frac{M}{R^2}.\ 2\pi R = \frac{M}{R}$

Now letting R→ ∞, We get that

$$|f^{I}(a)| = 0$$

i.e (a) = 0

Since "a" is arbitrary f(z) is a constant function….

7) Polynimials

Def:- (1) If P(z) & Q(z) are the two Polynimials then

$R(z) = \frac{P(z)}{Q(z)}$ is a Rational Function…. We assume P(z) & Q(z) have no commom factor and hence no common Zeros…

If a is Zero of Q(z), we Define R(z) = ∞, We can consider R(z) as a function with the values in the extended complex plane and as such it is continuous

(2) The Zeros of Q(z) are called Poles of R(z)

The order of Pole of R(z) is defined to be the order of the corresponding Zeros of Q(z)

/ A pole of order One is called a Simple Pole

/ All Zeros of R(z) are given by the Zeros of P(z)

Important :- 1) Zeros of R(z) are given by Roots of P(z)

2) Zeros of Q(z) are called Poles of R(z)

3) Poles of R(z) are the Zeros of Q(z)

8)LUCA'S THEOREM:-

If all zeros of a polynomial P(z) lie in a half plane, then all Zeros of the Derivative $P^{I}(z)$ lie in the same half plane…

Proof:- Let $P(z) = a_0 + a_1 z + a_2 z^2 + \text{--------} + a_n z^n$ be a Polynomial of degree n>0, with the zeros, say $a_1, a_2, a_3, \text{-------}, a_n$.

Then we can write P(z) as :-

$$P(z) = a_n (z-a_1)(z-a_2)\text{------------}(z-a_n)$$

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Formal Logarithmic Differential leads to :-

$$\frac{P'(z)}{(z)} = \frac{1}{z-a1} + \frac{1}{z-a2} + \text{----------------} + \frac{1}{z-an}$$

$$= \sum_{k=1}^{n} \frac{1}{z-ak}$$

Now Let H= $\{z: Im\,[\frac{z-a}{b}] < 0\}$ be the Right halp plane that contains all the Zeros of P(z), namely a_1, a_2, a_3, -------a_n...

Note that H is determined by the line

Z= a+bt

Since a_k 's lies in H, we have

$$I_m\,[\frac{a_k - a}{b}] < 0,\ k = 1,2,3,4,\ldots,\ n \text{ ------------------- (1)}$$

$$\&\ -I_m\,[\frac{a_k - a}{b}] > 0,\ k=1,2,3,4,\ldots,\ n$$

We prove the theorem by Showing That any $z_0 \notin H$ can never be a Zero of P(z), so that all the Zeros of (z), if any , should lie within H.

So let $z_0 \notin$ H.

$$\therefore\ I_m\,[\frac{z0-a}{b}] \geq 0 \text{ ------------------------------------(2)}$$

Now $\frac{z-ak}{b} = \frac{z0-a+a-ak}{b} = \frac{z0-a}{b} - \frac{ak-a}{b}$

$$\Rightarrow\ Im[\frac{z0-ak}{b}] = Im\,[\frac{z0-a}{b}] - Im[\frac{ak-a}{b}] > 0., \quad \text{from (1) \& (2)}$$

$$\text{Hence}\ \ Im\,[\frac{b}{z0-ak}] < 0$$

$$\text{Now}\ \frac{(Z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z-ak}$$

$$\longrightarrow \frac{(z0)}{(Z0)} = \sum_{k=1}^{n} \frac{b}{z-ak} < 0$$

$$\therefore \frac{(z0)}{(Z0)} \neq 0. \qquad \therefore P'(z0) \neq 0$$

Thus $z_0 \notin$ H implies that z_0 is not a Zero of $P'(z)$...

Hence all the Zeros of $P'(z)$ also lie in H.

Hence Proved

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