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An Innovative Method to the Fractional Diffusion-Wave Equation

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Abstract*: A fundamental mathematical model with numerous applications in physics, engineering, and biology is the fractional diffusion-wave equation. In this study, we suggest a novel and effective numerical approach for resolving this difficult problem. The fractional diffusion-wave equation has traditionally been computationally challenging and frequently restricted to particular situations or too simplistic scenarios. To overcome this issue, our revolutionary strategy makes use of cutting-edge numerical approaches and creative algorithms. We present a fresh discretization approach that preserves computing effectiveness while capturing the subtleties of fractional derivatives. The solutions to the linear and non-linear fractional diffusion wave equations have been found in this study using a new, creative technique. To show the effectiveness of the method, instances with clear solutions are applied.*

Keywords: New Iterative Method, Fractional Diffusion – wave Equation, Fractional initial value problems, Caputo fractional derivative

I. INTRODUCTION

A crucial mathematical framework that appears in many fields of science and engineering, the fractional diffusion-wave equation describes processes where wave propagation and diffusion take place simultaneously. It has a wide range of applications, including modeling biological events with anomalous diffusion, simulating electromagnetic wave propagation in complex media, and heat conduction in fractal media. The fractional derivatives in the equation, which include non-local and memory-dependent effects, make it difficult to solve the fractional diffusion-wave equation despite its significance. Significant efforts have been made in recent years to develop effective and precise numerical algorithms to solve this problem. Traditional numerical approaches that need high spatial and temporal resolutions, such as finite difference or finite element methods, result in computationally costly simulations and are frequently not suitable for handling fractional derivatives. Therefore, it is crucial to create novel approaches that can balance the conflicting requirements of accuracy and computing economy. The fractional diffusion-wave equation poses computational difficulties, which are addressed by an original approach in this study. Our methodology uses powerful numerical algorithms and ideas from fractional calculus to offer a fresh and effective approach to problem-solving. We accomplish a large decrease in computing cost while retaining a high degree of accuracy by redefining the discretization of fractional derivatives and using specialized numerical algorithms. This advancement makes it possible for scientists and engineers to investigate a larger range of uses for the fractional diffusion-wave equation, such as recreating intricate physical events and improving engineering designs.

The classical diffusion equation is converted into a space-time fractional diffusion wave equation by substituting a fractional derivative of order $\beta(1 < \beta \leq 2)$ for the second order space derivative and a fractional derivative of order $\propto (1 < \propto \leq 1)$ for the first order time derivative [2]. The literature has covered similar generalizations of the classical wave equation. Numerous academics have addressed the Riemann-Liouville derivative [3], [11], the Caputo derivative

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[4], [5], [9], [14], [15], and the Grunwald-Letnikov derivative [16] in the context of the diffusion-wave equations. The following equation's solution has been shown by Fujita [3] to exist and be unique.

$$D_t^{\alpha} u = D_x^{\beta} u, 0 < \alpha \leq 1, 0 < \beta \leq 2 \quad (1)$$

The time-fractional diffusion equation $(\beta = 2)$ reflects sub-diffusion for $(0 < \alpha < 1)$ according to Schneider and Wyss's research [4]. The increased diffusion represented by $(0 < \alpha < 2)$ only exists in one dimension, it has also been discovered.

In higher dimensions for $(\alpha > 1)$ the solutions do not have to be non-negative and cannot reflect any type of physical diffusion [4], [5].

In various areas of science and engineering, the fractional diffusion-wave equation is often utilized. A non-Markovian diffusion process with memory [9], charge transport in amorphous semiconductors [10], mechanical wave propagation in viscoelastic medium [6, 7, 8], and many more phenomena are all represented by these equations. In order to understand how mechanical diffusive waves move through viscoelastic media that experience power-law creep, Mainardi et al. [6, 7, 8] examined the fractional wave equation. To simulate electromagnetic, acoustic, and mechanical reactions, Nigmatullin [11] employed the fractional diffusion-wave equation. A continuous time random walk on fractals was studied by Roman and Alemany [12]. Giona, Cerbelli, and Roman [13] used fractional diffusion equations to study the relaxation processes in complicated viscoelastic material.

These equations have been solved using a number of different methods, including Green's function method [4], Finite sine transform method [14], Method of Images and Fourier transform [9], Separation of Variables method [15], Finite Difference method [16], and Adomian decomposition method (ADM) [17], [18]. In order to solve nonlinear functional equations, Daftardar-Gejji and Jafari [1] created the New Iterative Method (NIM). This approach has a pretty basic algorithm and is free of rounding off mistakes because it does not use discontinuity. Additionally, prior knowledge of ideas like homotopy or Lagrange multipliers is not necessary.

The following fractional diffusion-wave equation is solved using the NIM [1] in this article.

$$D_t^{\alpha} u(\bar{x},t) = \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u(\bar{x},t) + A(u(\bar{x},t)), 1 < \beta_i \leq 2, \quad (2)$$

Where $\bar{x} = (x_1, x_2, \dots \dots x_n) \in R^n$, a_i are constants, $A(u)$ is non- linear function of u, D_t^{α} and $D_{x_i}^{\beta_i}$ denote Caputo partial fractional derivatives with respect to t and with respect to x_i respectively.

II. PRELIMINARIES

In this portion, we set up the notations and review some fundamental definitions.

Definition1.1. A real function $f(x), x > 0$ is said to be in space $C_{\alpha}, \alpha \in \Re$, if there exists a real number $(> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C\,[0,\infty)$.

Definition1.2. A real $f(x), x > 0$ function **is** said to be in space C_{α}^{m}, $m \in \mathrm{IN} \cup \{0\}$, if $f^m \in C_{\alpha}$.

Definition1.3. Let $f \in C_{\alpha}$ and $\alpha \geq -1$ then the (Left sided) Riemann- Liouville integral of order μ, $\mu > 0$ is given by

$$I_t^{\mu} f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(x,\tau) d\tau, t > 0 \quad (3)$$

Definition1.4. The (left sided) Caputo partial fractional derivative of f with respect to , $f \in C_{-1}^{m}, m \in IN \cup \{0\}$, is defined as:

$$D_t^{\mu} f(x,t) = \frac{\partial^m}{\partial t^m} f(x,t), \mu = m$$
$$= I_t^{m-\mu} \frac{\partial^m f(x,t)}{\partial t^m}, m-1 < \mu < m, m \in \mathbb{N}. \quad (4)$$

Note that

$$I_t^{\mu} D_t^{\mu} f(x,t) = f(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k f}{\partial t^k}(x,0) \frac{t^k}{k!}, m-1 < \mu \leq m, m \in \mathbb{N}, \quad (5)$$

$$I_t^{\mu} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu} \quad (6)$$

III. THE NEW ITERATIVE METHOD

To solve the functional equation, Daftardar- Gejji and Jafari [1] have developed a novel iterative approach.

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$u(\bar{x},t) = f(\bar{x},t) + L(u(\bar{x},t)) + N(u(\bar{x},t))$ (7)

Where is a given function, L and N are given linear and non-linear functions of u respectively, We are looking for a solution u of having the series form:

$u(\bar{x},t) = \sum_{i=0}^{\infty} u_i(\bar{x},t)$ (8)

Since L is linear

$L(\sum_{i=0}^{\infty} u_i) = \sum_{i=0}^{\infty} L(u_i)$ (9)

The nonlinear operator N is decomposed..

$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty} \{N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)\}$ (10)

From equations (8), (10) and (7) is equivalent to

$\sum_{i=0}^{\infty} u_i = \mathrm{f} + \sum_{i=0}^{\infty} L(u_i) + N(u_0) + \sum_{i=1}^{\infty} \{N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j)\}$ (11)

We define the recurrence relation.

$$u_0 = \mathrm{f}$$

$$u_1 = \mathrm{L}(u_0) + \mathrm{N}(u_0)$$

$u_{m+1} = \mathrm{L}(u_m) + \mathrm{N}(u_0 + \cdots .... + u_m) - \mathrm{N}(u_0 + \cdots .... + u_{m-1}), \mathrm{m} = 1,2,3,..$ (12)

Then

$u_1 + \cdots .... + u_{m+1} = \mathrm{L}(u_0 + \cdots .... + u_m) + \mathrm{N}(u_0 + \cdots .... + u_m), \mathrm{m} = 1,2,3,4,..$ (13)

And

$\sum_{i=0}^{\infty} u_i = \mathrm{f} + \mathrm{L}(\sum_{j=0}^{i} u_i) + N(\sum_{j=0}^{i-1} u_i)$ (14)

The k term approximate solution of (7) - (8) is given by $u = u_0 + u_1 + \cdots + u_{k-1.}$ For the convergence of the method, we refer the reader to paper.

IV. FRACTIONAL INITIAL VALUE

We consider the following fractional initial value problem, for $\bar{x} \in \Re^n$

$D_t^{\alpha} u(\bar{x},t) = \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u(\bar{x},t) + A(u(\bar{x},t)), t > 0, m-1 < \alpha \leq m,$ (15)

$\frac{\partial^j u}{\partial t^j}(\bar{x},0) = h_j(\bar{x}), 0 \leq j \leq m-1, m = 1,2, 1 < \beta_i \leq 2$ (16)

Where a_i are constants, $A(u)$ is non-linear function of u and $h_k$are functions of $\bar{x}$. Applying I_t^{α} on both sides of (15) and using (16) we get.

$u(\bar{x},t) = \sum_{j=0}^{m-1} h_j(\bar{x}) \frac{t^j}{j!} + I_t^{\alpha}\left(\sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u(\bar{x},t)\right) + I_t^{\alpha} A(u).$ (17)

Equation (17) has the form (7) with $f = \sum_{j=0}^{m-1} h_j\left(\bar{x}\right)\frac{t^j}{j!}$, $\mathrm{L(u)} = I_t^{\alpha}\left(\sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u\right)$ and $\mathrm{N(u)} = I_t^{\alpha}\mathrm{A(u)}$ and can be solved using NIM.

V. ILLUSTRATIVE EXAMPLE

Some illustrative example is presented below.

Example1. Consider the time-fractional diffusion equation

$D_t^{\alpha} u(x,t) = u_{xx}(x,t), t > 0, x \epsilon \Re, 0 < \propto \leq 1,$ (18)

$u(x,0) = \sin(x)$ (19)

$u = \sin(x) + I_t^{\alpha} u_{xx}$ (20)

Using the algorithm (12) of NIM, we get the recurrence relation

$u_0 = \sin(x), u_1 = -\sin(\mathrm{x})\frac{\mathrm{t}^{\propto}}{\Gamma(\alpha+1)}, ...$ (21)

In general $u_j = (-1)^j \sin(x) \frac{\mathrm{t}^{j\propto}}{\Gamma(\mathrm{j}\alpha+1)}$ $, j = 0,1,2,3 ...$ The solution of (18) - (19) is thus

$$u(x,t) = \sum_{j=0}^{\infty} \mathrm{u_j(x,t)} = \mathrm{Sin(x)} \sum_{j=0}^{\infty} \frac{(-\mathrm{t}^{\alpha})^{\mathrm{j}}}{\Gamma(\mathrm{j}\alpha + 1)} = \mathrm{Sin(x)E_{\alpha}(-t^{\alpha})}$$

Example 2. Consider the time- fractional wave equation

$$D_t^{\alpha} u(x,t) = k \cdot u_{xx}(x,t), t > 0, x \epsilon \Re, 0 < \propto \leq 2, \quad (22)$$

$$u(x,0) = \mathrm{x}^2 \quad (23)$$

$$u_t(x,0) = 0$$

We get the equivalent integral equation of initial value problem (22) - (23) as

$$u = \mathrm{x}^2 + \mathrm{k} \cdot I_t^{\alpha} \mathrm{u_{xx}} \quad (24)$$

Applying the NIM, we get $u_0 = \mathrm{x}^2, u_1 = 2\mathrm{k} \cdot \frac{\mathrm{t}^{\propto}}{\Gamma(\alpha+1)}, u_2 = 0, \ldots.$

The solution of (22) - (23) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i = \mathrm{x}^2 + 2\mathrm{k} \cdot \frac{\mathrm{t}^{\propto}}{\Gamma(\alpha+1)} \quad (25)$$

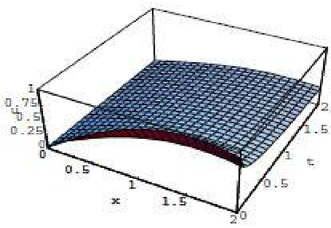

Fig. 1: (Ex. 1, $\alpha = 0.5$)

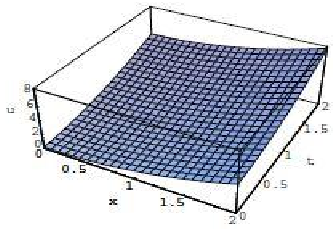

Fig. 2: (Ex. 2, $k = 1$, $\alpha = 1.7$)

Example 3. Consider the space-fractional diffusion equation

$$u_t(x,t) = k \cdot D_x^{\beta} u(x,t), t > 0, x \epsilon \Re, 0 < \beta \leq 2 \quad (26)$$

$$u(x,0) = \frac{2\mathrm{x}^{\beta}}{\Gamma(1+\beta)} \quad (27)$$

Integrating (26) and using (27) we get

$$u(x,t) = \frac{2\mathrm{x}^{\beta}}{\Gamma(1+\beta)} + \mathrm{k} \int_0^{\mathrm{t}} D_x^{\beta} u(x,t) dt \quad (28)$$

Applying the NIM, we get

$$u_0 = \frac{2\mathrm{x}^{\beta}}{\Gamma(1+\beta)}, u_1 = 2\mathrm{kt}, u_2 = 0, \ldots \quad (29)$$

$$\mathrm{T}u(x,t) = \frac{2\mathrm{x}^{\beta}}{\Gamma(1+\beta)} + 2\mathrm{kt} \quad (30)$$

Example 4. Now we consider the space and time fractional diffusion equation

$$D_t^{\alpha} u(x,t) = k \cdot D_x^{\beta} u(x,t), t > 0, x \epsilon \Re, \quad (31)$$

$$u(x,0) = \frac{3\mathrm{x}^{\beta}}{\Gamma(1+\beta)}, 0 < \propto \leq 1, 0 < \beta \leq 2 \quad (32)$$

Applying I_t^{α} on both sides of (31) and using condition (32), we get

$$u(x,t) = \frac{3\mathrm{x}^{\beta}}{\Gamma(1+\beta)} + I_t^{\alpha}(D_x^{\beta} u(x,t) \quad (33)$$

Using the algorithm of NIM we get

$$u_0 = \frac{3\mathrm{x}^{\beta}}{\Gamma(1+\beta)}, u_1 = 3\mathrm{k} \frac{\mathrm{t}^{\propto}}{\Gamma(\alpha+1)}, u_2 = 0, \ldots \quad (34)$$

Thus $u(x,t) = \frac{3\mathrm{x}^{\beta}}{\Gamma(1+\beta)} + 3\mathrm{k} \frac{\mathrm{t}^{\propto}}{\Gamma(\alpha+1)}$ is solution of (31) - (32)

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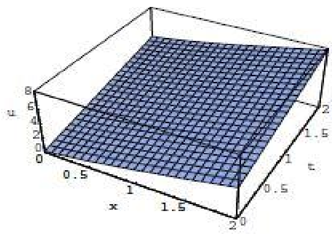

Fig. 3: (Ex. 3, $k = 1$, $\beta = 1.2$)

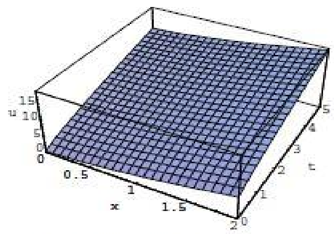

Fig. 4: (Ex. 4, $k = 1$, $\alpha = 0.8$)

VI. CONCLUSION

In this paper, we present an analytical approach for solving a fractional diffusion—wave equation with a reaction. We test the proposed approach by considering two numerical examples. We can summarize the novelty of this paper and future work in the following points: We solve two examples to show the efficiency of the proposed method. We notice that the proposed method gave the exact solutions in the first and second example after a few steps, which gives us numerical evidence that the proposed method is efficient. Based on this study, we think that the suggested approach is promising and suitable for other physical problems such as fractional kinetics and anomalous transport.

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