IJARSCT
International Journal of Advanced Research in Science, Communication and Technology
International Open-Access, Double-Blind, Peer-Reviewed, Refereed, Multidisciplinary Online Journal
ISSN: 2581-9429
Volume 3, Issue 1, September 2023
Impact Factor: 7.301

Applications of Spectral Sequences in Computing Derived Functors in Homological Algebra

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Abstract: *Spectral sequences are a fundamental computational tool in homological algebra, offering a systematic method to compute derived functors such as Ext and Tor, which are central to the study of modules, sheaves, and chain complexes. This review explores the theoretical background of spectral sequences, their construction from filtered complexes and double complexes, and highlights key applications in the computation of derived functors. Examples include the use of Grothendieck spectral sequences, Cartan–Eilenberg spectral sequences, and spectral sequences arising from exact couples.*

Keywords*:* Spectral sequences, derived functors, homological algebra

I. INTRODUCTION

Derived functors are indispensable in homological algebra for measuring the failure of exactness of functors. Common examples include $\mathrm{Ext}^n_R(M,N)$ and $\mathrm{Tor}^R_n(M,N)$
for R-modules M and N. Direct computation of these functors is often challenging, especially for complex filtrations or nontrivial extensions. Spectral sequences, introduced by Leray and further developed by Cartan and Eilenberg, provide an iterative computational framework to approach these problems (Weibel, 1994).

A spectral sequence is typically presented as a collection of pages (E_r,d_r), where each E_r is a bigraded module and $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is a differential satisfying d_r^2=0. The sequence converges to the graded pieces of the target homology or cohomology:

$$E_r^{p,q} \implies H^{p+q}(C)$$

Where C is a complex associated with the filtered or double complex structure.

SPECTRAL SEQUENCES FROM FILTERED COMPLEXES

Spectral sequences arising from filtered complexes are one of the most fundamental tools in homological algebra, providing a systematic framework for computing homology or cohomology of complex algebraic structures by breaking them into manageable stages. Consider a chain complex $C\cdot=\{C_n,d_n\}$ over a ring R, equipped with an increasing filtration F of sub complexes

$$0 = F_{-1}C^\bullet \subseteq F_0C^\bullet \subseteq F_1C^\bullet \subseteq \cdots \subseteq F_nC^\bullet = C^\bullet$$

where each $F_pC\cdot$ · is itself a subcomplex of C· This filtration allows one to study the complex incrementally by analyzing successive "layers" of the filtration. The associated graded complex is defined as

$$\mathrm{Gr}^F_p(C^\bullet) = F_pC^\bullet / F_{p-1}C^\bullet$$

And captures the structure of the successive quotients in the filtration. The key idea of spectral sequences is that instead of attempting to compute the homology of the entire complex Hn(C·) at once, one can iteratively compute homology on these graded pieces, propagating information through successive approximations called "pages" Er of the spectral sequence. The first page of the spectral sequence, denoted $E_0^{p,q}$,is obtained from the graded complex as

$$E_0^{p,q} = \mathrm{Gr}_p^F(C^{p+q}) = F_pC^{p+q}/F_{p-1}C^{p+q}$$

With differential $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ induced from the differential of the complex C· The homology of this page gives the first derived page

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_p^F(C^\bullet))$$

Which captures the initial approximation of the total homology by analyzing the layers individually? Higher differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ propagate corrections that account for interactions between different layers, gradually refining the approximation. As r→∞ the spectral sequence stabilizes, and the limiting page $E_\infty^{p,q}$ satisfies $E_\infty^{p,q} \cong \mathrm{Gr}_p^F H^{p+q}(C^\bullet)$ thus reconstructing the homology of the original complex from the filtered pieces. This entire construction can be summarized in a single unifying formula capturing the essence of the spectral sequence arising from a filtered complex:

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_p^F(C^\bullet)) \implies H^{p+q}(C^\bullet), \quad E_\infty^{p,q} \cong \mathrm{Gr}_p^F H^{p+q}(C^\bullet)$$

His formula embodies the dual role of spectral sequences: it both decomposes the problem into simpler homology computations and ensures convergence to the desired total homology. The strength of this method lies in its ability to handle highly intricate complexes that are otherwise intractable. For example, in the study of filtered chain complexes arising from simplicial or cellular decompositions in topology, one can associate a filtration by skeletal sub complexes, with F_pC^n being generated by n-chains in the p-skeleton. In such a case, the spectral sequence allows one to compute the homology of the entire space by iteratively examining lower-dimensional skeletons and their contributions to higher-dimensional homology.

Another important application arises in homological algebra over rings: given a module M with a filtration F by sub modules, the spectral sequence from the filtered complex associated with a projective resolution of M enables the computation of derived factors such as Ext and Tor in stages, starting from the graded components of the resolution. This method is particularly powerful when modules or complexes possess naturally compatible filtrations, for instance, in the study of chain complexes arising from tensor products, exact sequences, or sheaf co homology in algebraic geometry. Moreover, spectral sequences from filtered complexes can be linked with other constructions, such as double complexes, by taking horizontal or vertical filtrations, further expanding their computational scope.

The convergence of the spectral sequence ensures that while initial pages may only approximate the total homology, the process systematically refines the computation, correcting for higher-order interactions between filtration layers until the full homology emerges. Beyond purely computational advantages, filtered complex spectral sequences also provide deep theoretical insight: the associated graded pieces E1p, often reveal structural patterns and invariants of the underlying algebraic or topological objects, allowing algebraists and topologists to detect hidden relationships between sub complexes or sub modules.

For instance, in filtered differential graded algebras, the spectral sequence may identify torsion phenomena, extension problems, or obstructions that are otherwise invisible in the unfiltered complex. In practical computation, once the filtration and associated graded complex are defined, one constructs the initial page E0, computes its homology to obtain E1, and iteratively applies the differentials until stabilization. The final convergence formula

$$E_\infty^{p,q} \cong \mathrm{Gr}_p^F H^{p+q}(C^\bullet)$$

Guarantees that the process successfully reconstructs the total homology of C· while providing rich intermediate data at each stage. In summary, spectral sequences from filtered complexes transform the complex problem of computing homology into an organized, multi-stage process, with the single unifying formula

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_p^F(C^\bullet)) \implies H^{p+q}(C^\bullet), \quad E_\infty^{p,q} \cong \mathrm{Gr}_p^F H^{p+q}(C^\bullet)$$

Encapsulating the essence of the construction. This approach not only facilitates concrete computations in algebra and topology but also offers a conceptual framework to understand the layered structure of complexes, modules, and sheaves in modern homological algebra.

Consider a chain complex C· of R-modules with an increasing filtration F:

$$0 = F_{-1}C^\bullet \subseteq F_0C^\bullet \subseteq \cdots \subseteq F_nC^\bullet = C^\bullet$$

The associated graded complex is

$$\mathrm{Gr}_p^F(C^\bullet) = F_pC^\bullet / F_{p-1}C^\bullet$$

The spectral sequence $E_r^{p,q}$ arising from this filtration satisfies:

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}_p^F(C^\bullet)) \quad \text{and converges to} \quad H^{p+q}(C^\bullet)$$

This framework allows one to reduce complex computations of Hn(C·) to computations on simpler graded pieces, which is particularly useful in computing derived functors of composed functors (Mc Cleary, 2001).

GROTHENDIECK SPECTRAL SEQUENCE AND DERIVED FUNCTORS

The Grothendieck spectral sequence is one of the most powerful conceptual tools in homological algebra, providing a deep and systematic way to compute derived functors of a composite of functors by expressing them in terms of the derived functors of each component, and at its heart lies a single unifying formula, namely

$$E_2^{p,q} = R^pF\big(R^qG(-)\big) \implies R^{p+q}(F \circ G)(-),$$

Which compactly encodes a vast amount of structural and computational information? To appreciate the significance of this formula, one begins with the notion of derived functors, which arise naturally when a functor between abelian categories fails to be exact. Given a left exact functor, such as Hom, global sections, or invariants under a group action, its higher derived functors measure precisely how far exactness fails, capturing hidden extension and obstruction data that cannot be seen at the level of objects alone.

When two such functors G:A→B and F:B→C are composed, the naive expectation that the derived functors of F∘G should be obtained simply by composing the derived functors of F and G is generally false, and the Grothendieck spectral sequence is the refined mechanism that corrects this intuition. Under suitable hypotheses typically that A, B, C are abelian categories with enough injectives, that G sends injective objects of A to F-acyclic objects of B, and that both F and G are left exact the spectral sequence above exists and converges to the derived functors of the composite. Conceptually, the formula says that one should first resolve an object by injectives in A, apply G, then compute the derived functors of G, and finally apply the derived functors of F to the results, but that this process organizes itself not as a simple composition but as a multi-layered approximation whose successive stages are captured by the pages of a spectral sequence.

The E^2 -page, given explicitly by $R^pF(R^qG(-))$ is particularly important because it is often computable in concrete situations, while the abutment $R^{p+q}(F \circ G)(-)$ represents the ultimate target of interest. This single formula thus bridges local computations with global outcomes, revealing how homological complexity propagates through functorial composition. From a categorical perspective, the Grothendieck spectral sequence embodies the philosophy that derived categories and homological constructions are inherently hierarchical, and that information is best understood through filtrations and successive approximations rather than direct formulas.

Historically, Grothendieck introduced this spectral sequence in the context of sheaf cohomology, where it provides the theoretical foundation for many classical results, such as the Leray spectral sequence for a continuous map of topological spaces or a morphism of schemes, which itself is a special case of the general formula with G as the direct image functor and F as the global sections functor. In algebra, the same formalism explains relationships between Ext and Tor groups, group cohomology, and derived functors of invariants or coinvariants, again all subsumed under the single expression $E_2^{p,q} = R^pF(R^qG(-)) \Rightarrow R^{p+q}(F \circ G)(-)$. The power of this formula lies not merely in computation but in structure: it provides long exact sequences, filtration results, and vanishing criteria by analyzing the behavior of the E_2-terms, and it allows one to deduce properties of complicated derived functors from simpler building blocks. Moreover, it highlights a central theme of modern mathematics, namely that complex phenomena can often be understood by decomposing them into layered processes whose interactions are governed by universal principles.

In practice, many important theorems reduce to showing that certain $E_2^{p,q}$ terms vanish, forcing the spectral sequence to collapse and yielding direct isomorphisms between derived functors; in other cases, the differentials encode subtle extension data that reflects deep geometric or algebraic features of the objects under study. Thus, although the Grothendieck spectral sequence is often presented through elaborate constructions involving double complexes and filtrations, its essence is distilled into a single elegant formula that captures the relationship between composition and derivation, making it a cornerstone of homological algebra, algebraic geometry, and beyond.

Let F: A→B and G: B→C be left-exact functors between abelian categories, and assume F sends G-acyclic objects to G-acyclic objects. The Grothendieck spectral sequence computes the derived functors of the composition G∘F:

$$E_2^{p,q} = (R^pG)(R^qF)(A) \implies R^{p+q}(G \circ F)(A)$$

Where R^n denotes the n-th right derived functor. This spectral sequence is particularly useful in algebraic geometry and sheaf cohomology, as it allows the computation of derived functors in two stages (Weibel, 1994; Gelfand & Manin, 2003).

Example: For a sheaf F on a topological space X and an open cover U, one can compute $R^n\Gamma(X, \mathcal{F})$ using:

$$E_2^{p,q} = H^p(H^q(\mathcal{U}, \mathcal{F})) \implies H^{p+q}(X, \mathcal{F})$$

CARTAN–EILENBERG SPECTRAL SEQUENCE

The Cartan–Eilenberg spectral sequence is one of the most fundamental constructions in homological algebra, providing a powerful mechanism to compute the (co)homology of a total complex arising from a double complex, especially when derived functors are composed. At its core, the Cartan–Eilenberg spectral sequence can be encapsulated symbolically in a *single guiding formula* that expresses how homology is computed iteratively from simpler pieces, namely

$$E_2^{p,q} \cong R^pF(R^qG(A)) \implies R^{p+q}(F \circ G)(A)$$

And this compact expression captures the conceptual essence, computational purpose, and theoretical depth of the Cartan–Eilenberg spectral sequence in one unified statement. In this formula, A is an object of an abelian category, G and F are left exact functors between abelian categories with enough injectives, R_q^G denotes the q-th right derived functor of G, and R_p^F denotes the p-th right derived functor of F; the symbol ⟹ indicates convergence of the spectral sequence to the derived functors of the composite functor F∘GF. This single formula summarizes the entire philosophy of the Cartan–Eilenberg construction: rather than attempting to compute the derived functors of a composite functor directly, which is often difficult or impractical, one computes them step by step through an intermediate filtration encoded by the spectral sequence. Conceptually, the formula reflects the idea that homological information can be extracted layer by layer, first by resolving the object A to compute R_qG, and then by resolving those resulting objects to compute $R^pF(R^qG(A))$ with the spectral sequence organizing the interaction between these two homological processes.

From a structural perspective, this formula arises from a Cartan–Eilenberg resolution, which is a special type of double complex built from injective resolutions that are compatible in both directions. Starting with an object A, one first takes an injective resolution with respect to the functor G, and then each term in that resolution is further resolved injectively with respect to F. The resulting double complex Ip, has horizontal and vertical differentials corresponding to the

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ISSN: 2581-9429 Volume 3, Issue 1, September 2023 Impact Factor: 7.301

resolutions for F and G, respectively. The total complex Tot(I·,·) computes the derived functors of the composite F∘G, while the associated spectral sequence extracts information by filtering this total complex along one direction. The formula $E_2^{p,q} \cong R^pF(R^qG(A))$

The convergence statement $E_2^{p,q} \Longrightarrow R^{p+q}(F \circ G)(A)$ is equally significant; as it asserts that the spectral sequence stabilizes and reconstructs the desired derived functors in total degree $n = p + q.$ This convergence is typically guaranteed under standard boundedness or exactness conditions, such as when G sends injective objects to F-acyclic objects. Within the formula, convergence reflects a deep homological principle: although derived functors of composite functors are not, in general, simple compositions of derived functors, the obstruction to such simplicity is precisely measured by the higher differentials of the spectral sequence. Each differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ encodes hidden extension data, and the eventual collapse or stabilization of these differentials determines how the iterated derived functors assemble into the final derived functor of the composite. In this way, the single formula compactly represents both the approximation and the correction mechanism inherent in homological algebra.

Historically, this formula and the spectral sequence it represents were introduced by Henri Cartan and Samuel Eilenberg as part of their foundational work on homological algebra, particularly in their effort to systematize derived functors and cohomology theories. The Cartan–Eilenberg spectral sequence became a unifying tool connecting algebraic topology, algebraic geometry, and module theory, as it provides a general framework for understanding how complex homological invariants are built from simpler ones. For example, in group cohomology, the Lyndon–Hochschild–Serre spectral sequence is a special case of the Cartan–Eilenberg spectral sequence, and its governing principle can be traced directly back to the single formula above. Similarly, in sheaf cohomology, the Grothendieck spectral sequence, which computes the derived functors of a composite of two functors between categories of sheaves, is an explicit realization of the Cartan–Eilenberg framework expressed by this formula.

Philosophically, the importance of expressing the Cartan–Eilenberg spectral sequence in a single formula lies in its ability to condense a highly technical construction into a transparent conceptual statement. The formula highlights the layered nature of homological computation, the role of resolutions and exactness, and the manner in which global information emerges from local or intermediate data. It emphasizes that spectral sequences are not mysterious algebraic gadgets but are systematic bookkeeping devices that track how homology is assembled across multiple degrees and functorial steps. Thus, the expression

$$E_2^{p,q} \cong R^pF(R^qG(A)) \Longrightarrow R^{p+q}(F \circ G)(A)$$

Serves not only as the defining formula of the Cartan–Eilenberg spectral sequence but also as a conceptual roadmap for understanding its construction, convergence, and applications across modern mathematics.

Given a double complex C·, with horizontal and vertical differentials 'd′ and d′′, one can form total complexes Tot $\mathrm{Tot}(C^{\bullet,\bullet})$and associated spectral sequences. The Cartan–Eilenberg spectral sequence is:

$$E_1^{p,q} = H^q(C^{p,\bullet}, d'') \quad \text{or} \quad E_1^{p,q} = H^p(C^{\bullet,q}, d')$$

Converging to $H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$This method is widely used to compute Tor and Ext groups for complexes of modules or sheaves (Cartan & Eilenberg, 1956).

Equation Example for Tor:

$$\mathrm{Tor}_n^R(M, N) \cong H_n(\mathrm{Tot}(P_\bullet \otimes Q_\bullet))$$

Where P.→MP are projective resolutions and the associated spectral sequence gives:

$$E_{p,q}^2 = \mathrm{Tor}_p^R(H_q(Q_\bullet), M) \implies \mathrm{Tor}_{p+q}^R(M, N)$$

Applications and Examples

Group Cohomology: Hochschild–Serre spectral sequence computes Hn(G,M) from a normal subgroup GN⊴G:

$$E_2^{p,q} = H^p(G/N, H^q(N, M)) \implies H^{p+q}(G, M)$$

Algebraic Geometry: Derived functors of pushforward and pullback of sheaves, e.g., Rnf∗(F), are effectively computed using Grothendieck spectral sequences.

Homological Algebra: Computation of Ext and Tor groups via double complex resolutions is streamlined using Cartan–Eilenberg spectral sequences.

These examples highlight the practical power of spectral sequences in reducing intricate computations into manageable iterative stages.

II. CONCLUSION

Spectral sequences provide a systematic framework to compute derived functors in homological algebra. They simplify complex computations by breaking them into successive approximations via pages of the sequence, allowing for concrete calculations of Ext, Tor, and other homological invariants. Future research continues to explore spectral sequences in higher categorical contexts and computational homological algebra.

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