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Extension of Some Polynomial Inequalities to the Polar Derivative and the Generalized Polar Derivative

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Abstract: *In this paper certain polynomial inequalities for the polar derivative and the generalized polar derivative with restricted zeros are given, which generalize and refine some well-known polynomial inequalities due to Lax, Turán, AlSaeedi, Rather, Ali, Shafi, and Dar and others.*

Keywords: Inequalities, restricted zeros, polynomials, polar derivative, generalized polar derivative.

I. INTRODUCTION

If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then it is well known Bernstein´s inequality [6], on the derivative of a polynomial, we have

$$\max_{|z|=1}\left|P^{'}(z)\right| \le n \max_{|z|=1}|P(z)|\,. (1.1)$$

This result is best possible and equality holding for a polynomial that has all zeros at the origin.

If the polynomial $P(z)$ of degree n not vanishing in $|z|<1$, then Erdös [8] conjectured and Lax [13] proved that

$$\max_{|z|=1}\left|P^{'}(z)\right| \le \frac{n}{2}\max_{|z|=1}|P(z)|\,. (1.2)$$

If we restrict ourselves to the class of polynomials which have all its zeros in $|z|\le 1$, then it was proved by Turán [20], that

$$\max_{|z|=1}\left|P^{'}(z)\right| \ge \frac{n}{2}\max_{|z|=1}|P(z)|\,. (1.3)$$

The inequalities (1.2) and (1.3) are also best possible and become equality for polynomials which have all its zeros on $|z|=1$.

Let$D_\alpha P(z)$denote the polar derivative of a polynomial of degree$n$with respect to a complex number α, then

$$D_\alpha P(z) = n\,P(z) + (\alpha - z)P^{'}(z), \text{(see [15])}$$

The polynomial $D_\alpha P(z)$ is of degree at most $(n-1)$, and it generalizes the ordinary $P^{'}(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha P(z)}{\alpha}\right] = P^{'}(z),$$

uniformly with respect z for $|z|\le R\,, R>0$.

For each positive integer n,let $\mathcal{P}_n$ denote the set of all polynomials of degree n over the field $\mathbb{C}$ of complex number, $\partial\mathcal{P}_n$ denote the collection of all monic polynomials in $\mathcal{P}_n$ and $\mathbb{R}_+^n$ be the set of all $n-$ tuples $\gamma=(\gamma_1,\gamma_2,\dots,\gamma_n)$ of non – negative real numbers (not all zeros) with

$$\gamma_1+\gamma_2+\cdots+\gamma_n = \Lambda\,.$$

Let $D_\alpha^\gamma[P](z)$ denote the generalized polar derivative of the polynomial $P(z)$ as

$D_\alpha^\gamma[P](z) = \Lambda\,P(z) + (\alpha - z)P^\gamma(z)$,where$\Lambda=\sum_{j=1}^{n}\gamma_j$, for all $\gamma\in\mathbb{R}_+^n$, (see [18]).

Noting that for $\gamma=(1,1,1,\dots,1), D_\alpha^\gamma[P](z)=D_\alpha P(z)$.

Zygmund [21] extended Bernstein´s inequality (1.1) to L^Pnorm as

$$\left[\int_0^{2\pi}\left|P^{'}(e^{i\theta})\right|^q d\theta\right]^{1/q} \le n\left[\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right]^{1/q}, (1.4)$$

for any polynomial $P(z)$ of degree n and for any $q\ge 1$.

Malik [14] obtained the L^Pextension of (1.3) due to Turán [20] by proving that, if $P(z)$ has all its zeros in $|z|\le 1$, then for any $q>0$,

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$$n\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right]^{1/q} \leq \left[\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right]^{1/q} \max_{|z|=1}\left|P^{'}(z)\right|. (1.5)$$

In this paper we will extend and generalize of inequalities (1.2), (1.3) and (1.5) to the class of polar derivative and generalized polar derivative of polynomials.

II. LEMMAS

In this section, we introduce somelemmas that we need to prove the following theorems .The first Lemma is due to Govil and Rahman [10].

Lemma 2.1. If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n , then on $|z|=1$,

$$\left|P^{'}(z)\right| + \left|q^{'}(z)\right| \leq n \max_{|z|=1}|P(z)| \text{, where } q(z)=z^n\overline{P(1/\bar{z})} . (2.1)$$

Lemma 2.2. If $P(z)=a_0+\sum_{v=\mu}^{n} a_v z^v$, $1\leq\mu\leq n$,is a polynomial of degree$n$ having all its zeros in$|z|\leq k$, $k\geq 1$, then

$$\max_{|z|=1}\left|P^{'}(z)\right| \geq \frac{n}{1+k^{\mu}}\max_{|z|=1}|P(z)|. (2.2)$$

This lemma is due to [1]. The following Lemma is due to [16].

Lemma 2.3. If $P(z)=\sum_{v=0}^{n} a_v z^v$is a polynomial of degree n , then for $R\geq 1$

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)| . (2.3)$$

Lemma 2.4. If $P(z)=\sum_{v=0}^{n} a_v z^v$is a polynomial of degree n having all its zeros in $|z|\leq k$, $k>0$, then

$$\max_{|z|=R}|P(z)| \geq R^s\left(\frac{R+k}{1+k}\right)\max_{|z|=1}|P(z)| \text{ , for } k>1 \text{ and } k<R<k^2 (2.4)$$

where s is the order of possible zeros of $P(z)$at $z=0$.This Lemma is due to Jain [12]. The following Lemma is due to Rahman and Schmeisser [17] (see also [11]).

Lemma 2.5. If $P(z)=\sum_{v=0}^{n} a_v z^v$is a polynomial of degree n which dose not vanish in $|z|<1$, then for every $R\geq 1$ and $q>0$, we have

$$\int_0^{2\pi}\left|P\left(R\,e^{i\theta}\right)\right|^q d\theta \leq \frac{\left\{\int_0^{2\pi}\left|1+R^n e^{i\theta}\right|^q d\theta\right\}}{\left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}}\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta . (2.5)$$

Next Lemma is a special case of a result due to Aziz and Rather [4, 5] .

Lemma 2.6. If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z|\leq 1$ and $(z)=z^n\overline{P(1/\bar{z})}$, then for $|z|=1$, $\left|q^{'}(z)\right|\leq\left|P^{'}(z)\right|$.

Lemma 2.7. If $P(z)$ is a polynomial of degree n and $q(z)=z^n\overline{P(1/\bar{z})}$, then for every $q>0$ and γ real

$$\int_0^{2\pi}\int_0^{2\pi}\left|q^{'}\left(e^{i\theta}\right)+e^{i\gamma}P^{'}\left(e^{i\theta}\right)\right|^q d\theta\, d\gamma \leq 2\pi\, n^q\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta . (2.6)$$

The above lemma proved by [5]. The next lemma proved by [9].

Lemma 2.8. If$p$and q are arbitrary positive real numbers such that $q\geq px$, where $x\geq 1$, and if γ is any real number such that $0\leq\gamma<2\pi$, then

$$\frac{q+py}{x+y} \leq \left|\frac{q+pe^{i\gamma}}{x+e^{i\gamma}}\right| \text{ , for each } y\geq 1 . (2.7)$$

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The following Lemmas is due to Rather el at. [18].

Lemma 2.9. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n , then for $|z| = 1$

$$|Q^{\gamma}(z)| = |\Lambda\, P(z) - z\, P^{\gamma}(z)| \text{and} |P^{\gamma}(z)| = |\Lambda\, Q(z) - z\, Q^{\gamma}(z)|\,, \text{where } Q(z) = z^n \overline{P(1/\bar{z})}\,.$$

Lemma 2.10. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n which dose not vanish in $|z| < k$, $k \geq 1$, then

$$|Q^{\gamma}(z)| \leq k\, |P^{\gamma}(z)|\,, \text{for} |z| = 1\,, \text{where } Q(z) = z^n \overline{P(1/\bar{z})}\,. \quad (2.8)$$

The following Lemma is due to [2].

Lemma 2.11. If $P(z) = \sum_{v=0}^{n} a_v z^v$is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^n$ and on $|z| = 1$

$$|D_\alpha\, P(z)| \geq (|\alpha| - k^n)\, |P'(z)| \quad (2.9)$$

III. MAIN RESULTS

In this section, some theorems have been established and proved.

Theorem 3.1. If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^\mu$

$$\max_{|z|=1} |D_\alpha P(z)| \geq \left(\frac{n}{1+k^\mu}\right) (|\alpha| - k^\mu) \max_{|z|=1} |P(z)|\,. \quad (3.1)$$

Proof. Let$Q(z) = z^n \overline{P(1/\bar{z})}$, then $|Q'(z)| = |n\, P(z) - z\, P'(z)|$ on $|z| = 1$, we have for $|z| = 1$

$$|D_\alpha P(z)| = |n\, P(z) + (\alpha - z) P'(z)| = |\alpha\, P'(z) + n\, P(z) - z P'(z)| \geq |\alpha\, P'(z)| - |Q'(z)|$$

Using Lemma 2.1 in above, we get

$$|D_\alpha P(z)| \geq |\alpha\, P'(z)| + |P'(z)| - n \max_{|z|=1} |P(z)| \quad (3.2)$$

Since $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$. From inequality (2.2), we have

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^\mu} \max_{|z|=1} |P(z)| \quad (3.3)$$

Now, using (3.3) in (3.2), we get

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left[\frac{(|\alpha|+1)}{1+k^\mu} - 1\right] \max_{|z|=1} |P(z)|$$

This implies

$$\max_{|z|=1} |D_\alpha P(z)| \geq \left(\frac{n}{1+k^\mu}\right) (|\alpha| - k^\mu) \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem.

Remark 3.1. If we divide both sides of inequality (3.1) by $|\alpha|$ and $|\alpha| \to \infty$,we get inequality (2.2).It is worth to remark that by involving $min_{|z|=k} |P(z)|$, the generalized version of Theorem 3.1 can be stated as follows.

Theorem 3.2. If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^\mu$

$$\max_{|z|=1} |D_\alpha P(z)| \geq \left(\frac{n}{1+k^\mu}\right) \left[(|\alpha| - k^\mu) \max_{|z|=1} |P(z)| + \frac{1}{k^{n-\mu}} (|\alpha| + 1) \min_{|z|=k} |P(z)|\right]. \quad (3.4)$$

Proof. Since$P(z)$has all its zeros in $|z| \leq k$, $k \geq 1$. If $P(z)$has a zero on $|z| = k$, then $m = 0$ and result follows Theorem 3.1, we suppose that $P(z)$has all its zeros in $|z| < k, k \geq 1$, so $m > 0$. From Rouche'sTheorem$P(z) - m\, \beta\, z^n$ has all its zeros in $|z| \leq k$, $k \geq 1$, for every $\beta \in \mathbb{R}$ or $\mathbb{C}$ with $|\beta| < \frac{1}{k^n}$. Now, applying Theorem 3.1 to the polynomial $P(z) - m\, \beta\, z^n$

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$$\max_{|z|=1}|D_\alpha\{P(z)-m\,\beta\,z^n\}|\geq\frac{n\,(|\alpha|-k^\mu)}{(1+k^\mu)}\max_{|z|=1}|P(z)-m\,\beta\,z^n|$$

$$\max_{|z|=1}|D_\alpha P(z)-\alpha\,n\,m\,\beta\,z^{n-1}|\geq\frac{n\,(|\alpha|-k^\mu)}{(1+k^\mu)}\,[\max_{|z|=1}|P(z)|-m\,|\beta|\,].\,(3.5)$$

Let z_0 be a point on the unit circle such that $\max_{|z|=1}|D_\alpha P(z)|=|D_\alpha P(z_0)|$ in (3.1).

$$|D_\alpha P(z_0)-\alpha\,n\,m\,\beta\,{z_0}^{n-1}|\geq\frac{n\,(|\alpha|-k^\mu)}{(1+k^\mu)}\,[\max_{|z|=1}|P(z)|-m\,|\beta|].\,(3.6)$$

We choose the argument of β in (3.6) such that

$$|D_\alpha P(z_0)-n\,\alpha\,m\,\beta {z_0}^{n-1}|=|D_\alpha P(z_0)|-n\,|\alpha|m|\beta|.$$

$$\max_{|z|=1}|D_\alpha P(z)|\geq\frac{n\,(|\alpha|-k^\mu)}{(1+k^\mu)}\max_{|z|=1}|P(z)|+n\,m\,|\beta|\left\{|\alpha|-\frac{(|\alpha|-k^\mu)}{1+k^\mu}\right\}.\,(3.7)$$

Letting $|\beta|\to\frac{1}{k^n}$ in inequality (3.7), we obtain

$$\max_{|z|=1}|D_\alpha P(z)|\geq\left(\frac{n}{1+k^\mu}\right)\left[(|\alpha|-k^\mu)\max_{|z|=1}|P(z)|+\frac{1}{k^{n-\mu}}(|\alpha|+1)\min_{|z|=k}|P(z)|\right].$$

Dividing both sides of inequality (3.4) by $|\alpha|$ and taking $|\alpha|\to\infty$, we get

Corollary 3.1. If $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v, 1\leq\mu\leq n$ is a polynomial of degree n having all its zeros in $|z|\leq k\,,k\geq 1$, then

$$\max_{|z|=1}|P'(z)|\geq\left(\frac{n}{1+k^\mu}\right)\left[\max_{|z|=1}|P(z)|+\frac{1}{k^{n-\mu}}\min_{|z|=k}|P(z)|\right].\,(3.8)$$

If we take $\mu=1$ in inequality (3.8), we get

Corollary 3.2. If $P(z)=\sum_{v=o}^{n}a_vz^v$ is a polynomial of degree n having all its zeros in $|z|\leq k, k\geq 1$, then

$$\max_{|z|=1}|P'(z)|\geq\left(\frac{n}{1+k}\right)\left[\max_{|z|=1}|P(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|P(z)|\right].\,(3.9)$$

Remark 3.2. For $k=1$ in inequality (3.9),we get

$\max_{|z|=1}|P'(z)|\geq\frac{n}{2}[\max_{|z|=1}|P(z)|+\min_{|z|=1}|P(z)|]$,whichis result due to [3].

We present integral inequalities concerning polynomial.

Theorem 2.3. If $P(z)=\sum_{v=o}^{n}a_vz^v$ is a polynomial of degree n has no zeros in $|z|<1$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq 1$ and $q\geq 1$

$$\left[\int_0^{2\pi}\left||D_\alpha P(e^{i\theta})|+n\,m\right|^q\,d\theta\right]^{1/q}\leq n\,(1+|\alpha|)C_q\left[\int_0^{2\pi}\left||P(e^{i\theta})|+m\right|^q\,d\theta\right]^{1/q},(3.10)$$

where $C_q=\left[\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\gamma}|^q\,d\gamma\right]^{-1/q}$.

Proof. Since $P(z)$ has no zeros in $|z|<1$. Supposing $Q(z)=z^n\overline{P(1/\bar{z})}\,,P(z)=z^n\overline{Q(1/\bar{z})}$. Form Rouche´s theorem the polynomial $F(z)=P(z)+\lambda\,m$ has no zeros in $|z|<1$.

If $G(z)=z^n\overline{F(1/\bar{z})}=Q(z)+\bar{\lambda}\,m\,z^n$, then apply Lemma 2.6, we have

$$|F'(z)|\leq|G'(z)|\,,\text{for}|z|=1$$

that is,

$$|P'(e^{i\theta})|\leq|Q'(e^{i\theta})+\bar{\lambda}\,n\,m\,e^{i(n-1)\theta}|\leq|Q'(e^{i\theta})|+|\bar{\lambda}\,n\,m|\,,\text{for}|z|=1\ (3.11)$$

for each θ $(0\leq\theta<2\pi)$,

$$n\,P(e^{i\theta})-e^{i\theta}P'(e^{i\theta})=e^{i(n-1)\theta}\overline{Q'(e^{i\theta})},$$

as well as for polar derivative

$$D_\alpha P(e^{i\theta})=n\,P(e^{i\theta})+(\alpha-e^{i\theta})P'(e^{i\theta})\,,$$

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where $\alpha \in \mathbb{C}$. Using these equalities, we obtain

$$D_\alpha P(e^{i\theta})| \leq |\, n\, P(e^{i\theta}) - e^{i\theta} P'(e^{i\theta})\,| + |\alpha||P'(e^{i\theta})| \; = |Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})| \,. (3.12)$$

Using Lemma 2.8, we obtain

take $p = |P'(e^{i\theta})|$, $q = |Q'(e^{i\theta})| + |\bar{\lambda}\, n\, m|$, $x = 1$ and $y = |\alpha| \geq 1$ if

$$[(|Q'(e^{i\theta})| + |\bar{\lambda} n\, m|) + |\alpha||P'(e^{i\theta})|](|1 + e^{i\gamma}|) \leq (1 + |\alpha|)\left[\left||Q'(e^{i\theta})| + |\bar{\lambda} n\, m| + e^{i\gamma}|P'(e^{i\theta})|\right|\right]$$

$$[|Q'(e^{i\theta})| + |\alpha||P'(e^{i\theta})| + |\bar{\lambda}\, n\, m|](|1 + e^{i\gamma}|) \leq (1 + |\alpha|)\left[\left||Q'(e^{i\theta})| + e^{i\gamma}|P'(e^{i\theta})| + |\bar{\lambda}\, n\, m|\right|\right] \; (3.13)$$

Since for each $q \geq 1$ and arbitrary $a\,, b \;\in \mathbb{C}$ the equality

$$\int_0^{2\pi} |a + e^{i\gamma}\, b\,|^q\, d\gamma = \int_0^{2\pi} ||a| + e^{i\gamma}\, |b|\,|^q\, d\gamma$$

, using (3.12) in (3.13) and letting $\lambda \to 1$, we get for each $q \geq 1$holds

$$\int_0^{2\pi} |\,1 + e^{i\gamma}|^q\, d\gamma \int_0^{2\pi} \left||D_\alpha P(e^{i\theta})| + n\, m\,\right|^q\, d\theta \;\leq (1 + |\alpha|)^q$$

$$\left[\int_0^{2\pi}\int_0^{2\pi} |Q'(e^{i\theta}) + \; e^{i\gamma} P'(e^{i\theta})|^q\, d\gamma\, d\theta + \int_0^{2\pi} |\, n\, m|^q\, d\theta\,\right]. (3.14)$$

Using Lemma 2.7 in (3.14), we get

$$\int_0^{2\pi} |\,1 + e^{i\gamma}|^q\, d\gamma \int_0^{2\pi} \left||D_\alpha P(e^{i\theta})| + n\, m\,\right|^q\, d\theta \;\leq (1 + |\alpha|)^q n^q\, 2\pi \int_0^{2\pi} \left||\,P(e^{i\theta})| + m\,\right|^q d\theta$$

For each $q \geq 1$,

$$\left[\int_0^{2\pi} \left||D_\alpha P(e^{i\theta})| + n\, m\,\right|^q\, d\theta\right]^{1/q} \leq n\,(1 + |\alpha|)C_q \left[\int_0^{2\pi} \left||\,P(e^{i\theta})| + m\,\right|^q\, d\theta\right]^{1/q}.$$

This completes the proof of the Theorem 3.3.

We use Holder's inequality for the polar derivative and the generalize polar derivative of polynomial having all zeros inside a disk.

Theorem 3.4. If $P(z) = \sum_{v=o}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \leq k\,, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k^n$,$q > 1, r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$ and $k < R < k^2$

$$n\,(|\alpha| - k^n)\, R^s \left(\frac{R + k}{1 + k}\right)\left[\int_0^{2\pi} |P(e^{i\theta})|^q\;\, d\theta\right]^{1/q}$$

$$\leq B_q \left\{\int_0^{2\pi} |1 + e^{i\theta}|^{qr}\; d\theta\right\}^{1/qr} \left\{\int_0^{2\pi} |D_\alpha P(Re^{i\theta})|^{qs}\; d\theta\right\}^{1/qs}, (3.15)$$

where $B_q = \frac{\left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^q\, d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q\, d\theta\right\}^{1/q}}$.

Proof. The polynomial $G(z) = P(kz)$has all its zeros in $|z| \leq 1$and $H(z)$has all its zeros in $|z| \geq 1, H(z) = \; z^n \overline{G(1/\bar{z})}$ will has all its zeros in $|z| \geq 1$.

$$|H'(z)| \leq |n\, H(z) - z\, H'(z)|\,, \text{for}|z| = 1. (3.16)$$

Also since $G(z)$has all its zeros in $|z| \leq 1$, by Gauss - Lucas theorem all the zeros of $G'(z)$ also lie in $|z| \leq 1$. This implies that the polynomial $z^{n-1}\, \overline{G'(1/\bar{z})} \equiv n\, H(z) - z\, H'(z)$ dose not vanish in $|z| < 1$. Therefore, it follows from (3.16) that the function

Let $w(z) = \frac{z\, H'(z)}{(n\, H(z) - z\, H'(z))}$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function$1 + w(z)$ is subordination to the function $1 + z$. Hence by a well-known property of subordination [11], we have for each $q > 0$,

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$$\int_0^{2\pi} |1 + w(e^{i\theta})|^q \, d\theta \leq \int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta \,. (3.17)$$

Now,

$$(1 + w(z)) = \frac{n\, H(z)}{(n\, H(z) - z\, H'(z))} \,. (3.18)$$

And

$$|G'(z)| = |z^{n-1}\, \overline{G'(1/\bar{z})}| = |n\, H(z) - z\, H'(z)| \,, \text{for} |z| = 1 \,. (3.19)$$

Therefore, by (3.18), for $|z| = 1$, we have

$$n\, |H(z)| = (|1 + w(z)|)(|n\, H(z) - z\, H'(z)|).$$

Which by employing (3.19), yields the following

$$n\, |H(z)| = (|1 + w(z)|)|G'(z)| \,, \text{for} |z| = 1 \,. (3.20)$$

From (3.17) and (3.20), we deduce, for $q > 0$, that

$$n^q \int_0^{2\pi} |H(e^{i\theta})|^q d\theta \leq \int_0^{2\pi} (|1 + e^{i\theta}|^q |G'(e^{i\theta})|^q)\, d\theta \,, \text{for} |z| = 1.$$

This gives with the help of Holder's inequality for $r > 1$, $s > 1$ with $r^{-1} + s^{-1} = 1$

$$n^q \int_0^{2\pi} |H(e^{i\theta})|^q d\theta \leq \left\{\int_0^{2\pi} |1 + e^{i\theta}|^{qr} \, d\theta\right\}^{1/r} \left\{\int_0^{2\pi} |G'(e^{i\theta})|^{qs} \, d\theta\right\}^{1/s} \quad (3.21)$$

Since $H(z)$is a polynomial of degree n which dose not vanish in $|z| < 1$, using Lemma 2.5 with $R = k \geq 1$ to $H(z)$, we obtain

$$\int_0^{2\pi} |H(k\, e^{i\theta})|^q \, d\theta \leq \frac{\left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^q \, d\theta\right\}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta\right\}} \int_0^{2\pi} |H(e^{i\theta})|^q \, d\theta$$

From the fact that $|H(k\, e^{i\theta})| = k^n |P(e^{i\theta})|$ it follows that

$$n^q k^{nq} \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \leq \frac{\left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^q \, d\theta\right\}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta\right\}} n^q \int_0^{2\pi} |H(e^{i\theta})|^q \, d\theta \,, k \geq 1 \quad (3.22)$$

Using inequality (3.21) in inequality (3.22), we get

$$n^q k^{nq} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \leq \frac{\left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^q \, d\theta\right\}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta\right\}} \left\{\int_0^{2\pi} |1 + e^{i\theta}|^{qr} d\theta\right\}^{1/r} \left\{\int_0^{2\pi} |G'(e^{i\theta})|^{qs} d\theta\right\}^{1/s} \quad (3.23)$$

From Lemma 2.3, we have

$$\max_{|z|=R\geq 1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|$$

Since $G(z) = P(kz)$, then $G'(z) = k\, P'(kz)$ which is of degree $(n-1)$, we get

$$\max_{|z|=1} |G'(z)| = k \max_{|z|=k} |P'(z)| \,.$$

Which on using inequality (2.3) implies

$$\max_{|z|=1} |G'(z)| \leq k^n \max_{|z|=1} |P'(z)| \,. (3.24)$$

Now of Lemma 2.4, we have

$$\max_{|z|=1} |P(z)| \leq \frac{1}{R^s} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |P(z)| \,, (3.25)$$

where s is the order of possible zeros of $P(z)$at $z = 0$. On applying (3.25) to the polynomial $P'(z)$, we obtain

$$\max_{|z|=1} |P'(z)| \leq \frac{1}{R^s} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |P'(z)| \quad (3.26)$$

Therefore, now using (3.26) in (3.24), we get

$$\max_{|z|=1} |G'(z)| \leq \frac{k^n}{R^s} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |P'(z)| \quad (3.27)$$

Appling Lemma 2.11 in (3.27), we get

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$$|G^{'}(z)| \leq \frac{k^n}{R^s(|\alpha|-k^n)}\left(\frac{1+k}{R+k}\right)|D_\alpha\, P(Rz)|\,,\text{for}|z|=1\ (3.28)$$

Using (3.28) in (3.23), we get

$$n^q k^{nq}\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta \leq \frac{\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q\ d\theta\right\}}{\left\{\int_0^{2\pi}|1+e^{i\theta}|^q\ d\theta\right\}}\left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^{qr} d\theta\right\}^{1/r}$$

$$\left(\frac{k^n}{R^s(|\alpha|-k^n)}\right)^q\left(\frac{1+k}{R+k}\right)^q\left\{\int_0^{2\pi}\left|D_\alpha P\left(Re^{i\theta}\right)\right|^{qs} d\theta\right\}^{1/s}.$$

Which is equivalent to (3.15) for $q>1$. This proves Theorem .

Theorem 3.5. If $P(z)=\sum_{v=o}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$, $q>0$ and $r>1$, $s>1$ with $r^{-1}+s^{-1}=1$

$$\Lambda(|\alpha|-k)\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q\ d\theta\right]^{1/q} \leq \left(\int_0^{2\pi}\left|1+ke^{i\theta}\right|^{qr} d\theta\right)^{1/qr}\left(\int_0^{2\pi}\left|D_\alpha^\gamma[P]\left(e^{i\theta}\right)\right|^{qs} d\theta\right)^{1/qs}\quad(3.29)$$

Proof. Since $P(z)$ is a polynomial of degree n , from Lemma 2.8, we have

$$|Q^\gamma(z)|=|\Lambda\, P(z)-z\, P^\gamma(z)|\text{and}|P^\gamma(z)|=|\Lambda\, Q(z)-z\, Q^\gamma(z)|\,,\text{for}|z|=1\ (3.30)$$

From Lemma 2.9, we have

$$|Q^\gamma(z)|\leq k\,|P^\gamma(z)|\,,\text{for}|z|=1\ (3.31)$$

Using (3.30) in (3.31), we get for $|z|=1$,

$$|Q^\gamma(z)|\leq k\,(|\Lambda\, Q(z)-z\, Q^\gamma(z)|)\ (3.32)$$

From (3.30) and (3.31) for every real or complex number α with $|\alpha|\geq k$and $|z|=1$, we have

$$\left|D_\alpha^\gamma[P](z)\right|\geq|\alpha||P^\gamma(z)|-(\,|\Lambda\, P(z)-z\, P^\gamma(z)|)=|\alpha||P^\gamma(z)|-|Q^\gamma(z)|\ \geq(|\alpha|-k)|P^\gamma(z)|(3.33)$$

Since $P(z)$has all its zeros in $|z|\leq k$, it follows from Rather el at. [19], we have

Every convex set containing all the zeros of $P(z)$ also contains the zeros of $P^\gamma(z)$for all $\gamma\in\mathbb{R}_+^n$ .Therefore, from (3.30) that the function

$$w(z)=\frac{z\, Q^\gamma(z)}{k\left(\Lambda\, Q(z)-z\, Q^\gamma(z)\right)}$$

is analytic for $|z|\leq 1$ and $|w(z)|\leq 1$ for $|z|=1$. Furthermore, $w(0)=0$. Thus the function$1+k\, w(z)$ is subordination to the function $1+k\, z$. Hence by a well-known property of subordination [11] ,we have for each $q>0$,

$$\int_0^{2\pi}\left|\,1+k\, w(e^{i\theta})\,\right|^q\ d\theta\leq\int_0^{2\pi}\left|\,1+ke^{i\theta}\right|^q\ d\theta\ (3.34)$$

Now

$$\left(1+k\, w(z)\right)=\frac{\Lambda\, Q(z)}{(\,\Lambda\, Q(z)-z\, Q^\gamma(z)\,)}$$

And$|P^\gamma(z)|=|\Lambda\, Q(z)-z\, Q^\gamma(z)|$, for $|z|=1$

therefore for $|z|=1$,

$\Lambda\,|Q(z)|=(|1+k\, w(z)|)(|\Lambda\, Q(z)-z\, Q^\gamma(z)|)=(|1+k\, w(z)|)|P^\gamma(z)|$

Equivalent,

$\Lambda\left|z^n\overline{P(1/\bar{z})}\right|=(\,|1+k\, w(z)|\,)\,|P^\gamma(z)|$. This implies

$$\Lambda\,|P(z)|=(|1+k\, w(z)|)\,|P^\gamma(z)|\,,\text{for}|z|=1.\ (3.35)$$

From (3.33) and (3.35), we deduce that for $q>0$,

$$\Lambda^q(|\alpha|-k)^q\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\leq\left(\int_0^{2\pi}\left|1+k\, w\left(e^{i\theta}\right)\right|^q\left|D_\alpha^\gamma[P]\left(e^{i\theta}\right)\right|^q\right)d\theta$$

This gives with the help of Holder's inequality and (3.34) for $r>1, s>1$ with $r^{-1}+s^{-1}=1$,

$$\Lambda^q(|\alpha|-k)^q\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\leq\left(\int_0^{2\pi}\left|1+k\, e^{i\theta}\right|^{qr} d\theta\right)^{1/r}\left(\int_0^{2\pi}\left|D_\alpha^\gamma[P]\left(e^{i\theta}\right)\right|^{qs}\ d\theta\right)^{1/s},$$

equivalent ,

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$$\Lambda(|\alpha|-k)\left[\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right]^{1/q} \leq \left(\int_0^{2\pi}\left|1+k\,e^{i\theta}\right|^{qr} d\theta\right)^{1/qr}\left(\int_0^{2\pi}\left|D_\alpha^\gamma[P](e^{i\theta})\right|^{qs} d\theta\right)^{1/qs}.$$

Which proves the desired result.

IV. CONCLUSION

We established some Turan-type derivatives inequalties whereas Zygmund-type and Malik-type integral inequalities for the polar derivative and generalize polar derivative of polynomials that are vanishing in a disk or not vanishing in disk.The obtained results sharpen and improvement some already known estimates that relate to polar derivative and integral inequalities for the polar derivative and generalize polar derivative of polynomials .

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