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Fixed Point Results in Rough Metric Spaces

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Abstract: *In the present paper, we prove some fixed point results in Rough metric spaces. Rough set theory has a significant importance in many fields. The aim of this paper is to describe some properties of rough sets. Moreover, we give further study on Pawlak rough sets and introduce new examples some rough set theory concepts.*

Keywords: Metric Spaces, Rough Sets, Rough Metric Spaces

I. INTRODUCTION

Pawlak introduced to the concept of rough sets during the early 1980's. It is an wonderful tool to handle the granularity of data. There are lots of conjoint applications of rough set theory and topology in different areas of science.

In 1982, Z. Pawlak [1] gave the concept of Rough sets. After that, R. Biswas and S. Nanda [2] proved Rough groups and Rough subgroups in 1994. Later, R. Biswas [3] defined Rough Metric spaces in 1996 and in 1999, Fixed point theorems in rough metric spaces [4]. In addition, H.X. Phu [5] extended Rough convergence in normed linear spaces in 2001. After that, B. Davvaz [6] obtained a Roughness in rings in 2004 and Roughness in modules in 2006. Z. Pawlak [7] presented Rough sets: Some extensions in 2007 and R. Bhardwaj *et al.*, [8] proved application of fixed point theory in metric spaces. In 2008, QingE Wu *et al.*, proved Topology theory on rough sets. Topological structures in rough set theory: A survey was introduced in 2019 by Pankaj Kumar Singh and Surabhi Tiwari [9].

In particular, S.T. Almohammdi and C.Ozel [10] generalized a new approach to rough vector spaces in 2019. Later, Ramakant Bhardwaj [11] presented Fixed point results in Compact rough metric spaces in 2022.

Some Basic Definitions

1.1 Definition: A metric space is a set X together with a function d(called a metric or distance function) which assigns a real number d(x,y) to every pair x,y ϵX satisfying the properties :

$d(x,y) \geq 0 \rightarrow x = y$

$d(x,y) = d(y,x)$

$d(x,y) + d(y,z) \leq d(x,z)$

1.2 Rough Set (Pawlak): Let $A_s = (U, R)$ be an approximation space and P(U) be the power set of U. Let $\approx A_s$, $\simeq A_s$, $\asymp A_s$, roughly bottom -equal , roughly top equal and roughly equal symbols. Then all three becomes equivalence relation on P(U). The following approximation spaces are defined upon any approximation space $A_s = (U,R)$:

$\underline{A}_s^* = (P(U), \approx A_s)$, $\overset{\frown}{A}_s^* = (P(U), \simeq A_s)$, $A_s^* = (P(U), \asymp A_s)$ which are known as the lower extension, upper extension and extension of the approximation space AS respectively, and the equivalence classes of $\asymp A_s$, $\approx A_s$, $\simeq A_s$ are called rough, rough lower and rough upper sets.

1.3 Rough Subset:- If $A_s = (U, R)$ is an approximation space and A, B are subsets of U, then A is said to be rough subset of B in A_s if approx A_s (A) ⊂ approx A_s (B) . In Symbol we denote it as A⊂ B.

II. MAIN RESULTS

In this section, we prove fixed point results in rough metric space. Our first new result is the following:

Theorem2.1:- Let (Y, d) be a rough metric space and let G be a nonempty closed subset of Y. Let U,V: G→ G be such that

$d(Ux, Vy) \leq \frac{1}{2}(d(Wx, Vy) + d(Wy, Ux) - \alpha(d(Wx, Vy), d(Wy, Ux))$............(1.1)

for every pair $(x,y) \in Y \times Y$; where $\alpha : [0,\infty)^2 \rightarrow [0, \infty)$ is a continuous function such that

$\alpha(x,y) = 0$ if and only if $x = y = 0$ and $W : G \rightarrow Y$ satisfying the following hypothesis

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(1)$UG \subseteq WG$ and $VG \subseteq WG$

(2)The pairs (U, W) and (V,W) are weakly compatible.

In addition, assume that W(G) is a closed subset of Y. Then, U and V and W have a unique common fixed point.

Proof:- Let $x_0 \in G$ be arbitrary. Using(1) there exist two sequence $\{x_n\}^{\infty}_{n=0}$ and $\{y_n\}^{\infty}_{n=0}$ such that $y_0 = Ux_0 = Wx_1$, $y_1 = Vx_1 = Wx_2$, $y_2 = Ux_2 = Wx_3$, ………., $y_{2n} = Ux_{2n} = Wx_{2n+1}$, $y_{2n+1} = Vx_{2n+1} = Wx_{2n+2}$,….

We complete the proof in three steps.

Step1:- We will prove that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Let n = 2k , using condition (1) , we obtain that

$d(y_{2k+1}, y_{2k}) = d(Ux_{2k}, Vx_{2k+1})$

$\leq \frac{1}{2}(d(Wx_{2k}, Vx_{2k+1}) + d(Wx_{2k+1}, Ux_{2k})) - \alpha(d(Wx_{2k}, Vx_{2k+1}), d(Wx_{2k+1}, Ux_{2k}))$

$= \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) - \alpha(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k}))$………(1.2)

$\leq \frac{1}{2} d(y_{2k-1}, y_{2k+1})$

$\leq \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1}))$

Hence , $d(y_{2k+1}, y_{2k}) \leq d(y_{2k}, y_{2k-1}))$

If n = 2k+1 , similarly we can prove that ,

$d(y_{2k+2}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k})$

Thus $d(y_{n+1}, y_n)$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Assume that, $\lim_{n\to\infty} d(y_{n+1}, y_n) = t$.

From the above argument we have

$d(y_{n+1}, y_n) \leq \frac{1}{2} d(y_{n-1}, y_{n+1})$

$\leq \frac{1}{2} d(y_{n-1}, y_n) + d(y_{n-1}, y_n)$ …………(1.3)

If $n \to \infty$, we have

$t \leq \lim_{n\to\infty} \frac{1}{2} d(y_{n-1}, y_{n+1}) \leq t$.

Therefore, $\lim_{n\to\infty} d(y_{n-1}, y_{n+1}) = 2t$.

We have proved in (1.2)

$d((y_{2k+1}, y_{2k}) = d(Ux_{2k}, Vx_{2k+1})$

$\leq \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) - \alpha(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k}))$………(1.4)

Now, if $k \to \infty$ and using the continuity of α we obtain

$t \leq \frac{1}{2}.2t - \alpha(2t, 0)$,

and consequently, $\alpha(2t, 0) = 0$. This gives us that

$t = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$…………..(1.5)

by our assumption about α.

Step 2:- $\{x_n\}$ is a Cauchy.

Since $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, it is sufficient to show that the subsequence $\{x_{2n}\}$is a Cauchy sequence.

Suppose that $\{x_{2n}\}$ is not a Cauchy sequence .Then there exist $\varepsilon > 0$ for which we can find subsequence $\{x_{2m(k)}\}$and $\{x_{2n(k)}\}$of $\{x_{2n}\}$ such that n(k) is the least index for which $n(k) > m(k) > k$ and $d((x_{2m(k)}, x_{2n(k)}) \geq \varepsilon$.

This means that $d((x_{2m(k)}, x_{2n(k)-2}) < \varepsilon$…………..(1.6)

From triangle inequality

$\varepsilon \leq d((x_{2m(k)}, x_{2n(k)}) \leq d((x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)})$

$\leq \varepsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)})$…………..(1.7)

Letting $k \to \infty$ and using (1.5) we can conclude that

$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \varepsilon$ ...……(1.8)

Moreover, we have

$| d((x_{2m(k)}, x_{2n(k)+1}) - d((x_{2m(k)}, x_{2n(k)}) | \leq d((x_{2n(k)}, x_{2n(k)+1})$…………(1.9)

And $| d((x_{2n(k)}, x_{2m(k)-1}) - d((x_{2n(k)}, x_{2m(k)}) | \leq d((x_{2m(k)}, x_{2m(k)-1})$………(1.10)

And $| d((x_{2n(k)}, x_{2m(k)-2}) - d((x_{2n(k)}, x_{2m(k)-1}) | \le d((x_{2m(k)-2}, x_{2m(k)-1}) \ldots\ldots\ldots(1.11)$

Using (1.5), (1.8), (1.9), (1.10) and (1.11) we get

$$\lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)}) = \lim_{k\to\infty} d(x_{2m(k)-1}, x_{2n(k)-1}) \ldots\ldots\ldots (1.12)$$

$$= \lim_{k\to\infty} d(x_{2m(k)-2}, x_{2n(k)}) = \varepsilon.$$

Now, from (1.1) we have

$d(x_{2m(k)-1}, x_{2n(k)}) = d(Uy_{2n(k)}, Vy_{2m(k)-1})$

$\le\frac{1}{2}(d(Wy_{2n(k)}, Vy_{2m(k)-1}) + d(Wy_{2m(k)-1}, Uy_{2n(k)})) - \alpha(d(Wy_{2n(k)}, Vy_{2m(k)-1}), d(Wy_{2m(k)-1}, Uy_{2n(k)}))$

$= \frac{1}{2}(d(x_{2n(k)-1}, x_{2m(k)-1}) + d(x_{2m(k)-2}, x_{2n(k)})) - \alpha(d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)}))$

$\le \frac{1}{2}(d(x_{2m(k)-1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)+1})) \ldots\ldots\ldots\ldots(1.13)$

If $k \to \infty$ in the above inequality, from (1.12) and the continuity of α, we have

$\varepsilon \le ½ (\varepsilon + \varepsilon) - \alpha(\varepsilon, \varepsilon)$

and from the last inequality $\alpha(\varepsilon, \varepsilon) = 0$. By our assumption about α, we have $\varepsilon = 0$ which is a contradiction.

Step 3:- U, V and W have a common fixed point.

Since (Y, d) is a complete and $\{x_n\}$ is Cauchy, there exist $z \in Y$ such that $\lim_{n\to\infty} x_n = z$. Since G is closed and $\{x_n\} \subseteq G$, we have $z \in G$. By assumption W(G) is closed, so there exist $t \in G$ such that $z = Wt$. For all $n \in N$

$d(Ut, x_{2n+1}) = d(Ut, Vy_{2n+1})$

$\le\frac{1}{2}(d(Wt, Vy_{2n+1}) + d(Wy_{2n+1}, Ut)) - \alpha(d(Wt, Vy_{2n+1}), d(Wy_{2n+1}, Ut)) \ldots\ldots\ldots\ldots(1.14)$

$= \frac{1}{2}(d(z, x_{2n+1}) + d(x_{2n}, Ut)) - \alpha(d(Wt, Vy_{2n+1}), d(Wy_{2n+1}, Ut))$

If $n \to \infty$,

$d(Ut, z) \le \frac{1}{2}(d(z, z) + d(z, Ut)) - \alpha(d(Wt, z), d(z, Ut))$

and hence $\alpha(0, d(z, Ut)) \le -1/2\, (d(Ut, z)) \le 0$,

therefore $d(z, Ut) = 0$. Therefore $Ut = z$.

Similarly, $Vt = z$. So $Ut = Vt = Wt = z$. Since the pairs (W, U) and (W, V) are weakly compatible, we have $Uz = Vz = Wz$. Now we can have

$d(Uz, x_{2n+1}) = d(Uz, Vy_{2n+1}) \le \frac{1}{2}(d(Wz, Vy_{2n+1}) + d(Wy_{2n+1}, Uz)) - \alpha(d(Wz, Vy_{2n+1}), d(Wy_{2n+1}, Uz)) \ldots\ldots\ldots(1.15)$

$= \frac{1}{2}(d(Wz, x_{2n+1}) + d(x_{2n}, Uz)) - \alpha(d(Wz, x_{2n+1}), d(x_{2n}, Uz))$

If $n \to \infty$, since $Uz = Vz = Wz$, we obtain

$d(Uz, z) = \frac{1}{2}(d(Uz, z) + d(z, Uz)) - \alpha(d(Uz, z), d(z, Uz)) \ldots\ldots\ldots\ldots(1.16)$

Hence, $\alpha(d(Uz, z), d(z, Uz)) = 0$ and so $d(Uz, z) = 0$.

Therefore $Uz = z$ and from $Uz = Vz = Wz$ we conclude that $Uz = Vz = Wz = z$.

Uniqueness of the common fixed point follows from (1).

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